# Episodic Liquidity Crises:

# The Effect of Predatory and Cooperative Trading

Bruce Ian Carlin[1] Miguel Sousa Lobo[2] S. Viswanathan[3]

November 22, 2004

[1]Fuqua School of Business, Duke University, bruce.carlin@duke.edu,
[2]Fuqua School of Business, Duke University, mlobo@duke.edu
[3]Fuqua School of Business, Duke University, viswanat@duke.edu

**Abstract**

We develop a theoretical model that explains how episodic illiquidity can result from a breakdown in cooperation between traders and manifest itself in predatory trading. In a multi-period framework, and with a continuous-time stage game with an asset-pricing equation that accounts for transaction costs, we describe an equilibrium where traders cooperate most of the time through repeated interaction and provide 'apparent liquidity' to each other. Cooperation can break down, especially when the stakes are high, and lead to predatory trading and episodic illiquidity. Equilibrium strategies involving cooperation across markets can cause the contagion of illiquidity.

Why is illiquidity rare and episodic? Pastor and Stambaugh (2003) detect only 14 aggregate low-liquidity months in the time period 1962-1999. The purpose of this paper is to develop a theoretical model that explains how episodic illiquidity can result from a breakdown in cooperation between traders and manifest itself in predatory trading.

We develop a dynamic model of trading based on liquidity needs. During each period, a *liquidity event* may occur in which a trader is required to liquidate a large block of an asset in a relatively short time period. This need for liquidity is observed by a tight oligopoly, whose members may choose to predate or cooperate. Predation involves racing and fading the *distressed* trader to the market, causing an adverse price impact for the trader[1]. Cooperation involves refraining from predation and allows the distressed trader to transact at more favorable prices. In our model, traders cooperate most of the time through repeated interaction, providing 'apparent liquidity' to each other. However, episodically this cooperation breaks down, especially when the stakes are high, leading to opportunism and loss of this apparent liquidity.

[1] Predatory trading has been defined by Brunnermeier and Pedersen (2004) as trading that induces and/or exploits another investor's need to change their position. It is important to distinguish predatory trading from front-running. Front-running is an illegal activity in which a specialist, acting as an agent of an investor, trades on his own account in the same direction as his client before he fulfills his client's order. In this way, the specialist profits but violates his legal obligation as an agent of the investor. Predatory activity occurs in the absence of such a legal obligation.

The following quote provides a recent example of an episodic breakdown in cooperation between cooperative periods in the European debt market (New York Times Sept. 15, 2004):

> "*...The bond sale, executed Aug. 2, caused widespread concern in Europe's markets. Citigroup sold 11 billion euros of European government debt within minutes, mainly through electronic trades, then bought some of it back at lower prices less than an hour later, rival traders say. Though the trades were not illegal, they angered other bond houses, which said the bank violated an unspoken agreement not to flood the market to drive down prices.*"

This suggests that market participants cooperate, though there is episodic predation which leads to acute changes in prices. It also points out that predatory behavior can involve either exploiting a distressed trader's needs or inducing another trader to be distressed.

Cooperation has been empirically demonstrated in the Interbank market. Cocco, Gomes, and Martins (2003) detect evidence that banks provide liquidity to each other in times of financial stress. They also find that banks establish lending relationships in this market to provide insurance against the risk of shortage or excess of funds during the reserve maintenance period. Cooperation and reputation have been documented to affect liquidity costs on the floor of the New York Stock Exchange (NYSE). Battalio, Ellul, and Jennings (2004) show an increase in liquidity costs in the trading days surrounding a stock's relocation on the floor of the exchange.[2] They find that brokers who simultaneously relocate with the stock and continue their long-term cooperation with the specialist obtain a lower cost of liquidity, which manifests in a smaller bid-ask spread.

In our *predatory stage game*, each trader faces a continuous-time, dynamic programming problem. Our model is closely related to the model by Brunnermeier and Pedersen (2004), except that we use a pricing equation that is linear in the first derivative of inventory of the asset. Thus, the price impact of trading is endogenous and results, roughly speaking, quadratic transaction costs. Also, in contrast to the Brunnermeier and Pedersen formulation, our model predicts price jumps during incidents of predation, which in turn allows us to model episodic illiquidity in the dynamic game. Further, we are able to derive the aggregate surplus losses to the traders when predatory behavior occurs.

In the equilibrium of our one-period model, traders 'race' to market, selling quickly in the beginning of the period. In the equilibrium strategy traders sell-off at a decreasing exponential rate. Also in equilibrium, predators initially race the distressed traders to market, but eventually 'fade' them and buy back. This racing and fading behavior is well-known in the trading industry and has been previously modelled by Foster and Viswanathan (1996). The associated trading volumes are also consistent with the U-shaped daily trading volume seen in financial markets.

We model cooperation by embedding the predatory stage game in a dynamic

[2]This is an exogenous event that changes long-run relationships between brokers and the specialist.

game. We first consider an infinitely-repeated game in which the magnitude of the liquidity event is deterministic and fixed. In this framework, we derive an equilibrium which is Pareto superior for the traders and describe how the cooperation between traders avoids the surplus loss due to predatory trading. We also model the interaction between strategic traders in this environment and outsiders who trade in the market.

We next model episodic illiquidity by allowing the magnitude of the liquidity event in the repeated game to be stochastic. Given stochastic liquidity shocks, we provide predictions as to the magnitude of liquidity event required to trigger liquidity crisis and demonstrate explicitly how a breakdown in cooperation leads to observed price volatility. Finally, we allow for multimarket contact in both the deterministic and stochastic versions of our repeated game. This provides for greater cooperation across markets, but leads to contagion of predation and liquidity crisis across all markets.

Several empirical implications emerge from our model. First, our model predicts that non-anonymous markets should be stable most of the time with high 'apparent' liquidity, but will experience illiquidity in an episodic fashion. This fact is consistent with the observed rareness of liquidity events in financial markets (Pastor and Stambaugh 2003; Gabaix, Krishnamurthy, and Vigneron 2004). Second, occasions where one player faces extreme financial distress are associated with periods of reduced liquidity. Finally, our model suggests that illiquidity is usually observed across markets (ie. contagion occurs), and not in isolation.

The paper is organized as follows. Section 1 introduces the pricing relationship. We provide closed-form solutions to the problems when one distressed trader liquidates a block of assets, when multiple traders liquidate, when there exists one opportunistic trader and one distressed trader, and when there exist multiple predators and distressed traders. Section 2 provides the solution to the supergame that fixes the magnitude of the liquidity event and uses the stage game with one predator and one distressed trader as a basis. Section 2 also models the trading relationship between insiders and outsiders in these markets. Section 3 models episodic illiquidity and the contagion of illiquidity across markets. Section 4 concludes.

## Related literature

There exists significant empirical evidence that illiquidity in the market is rare and episodic (Pastor and Stambaugh 2003; Gabaix, Krishnamurthy, and Vigneron 2004). As previously noted, Pastor and Stambaugh (2003) detect only 14 aggregate low-liquidity months in the time period 1962-1999. Likewise, Gabaix, Krishnamurthy, and Vigneron (2004) document only two episodes in the last decade during which the Mortgage-Treasury spread was excessively wide.

Other work on predatory trading includes papers by Brunnermeier and Pedersen (2004) and Attari, Mello, and Ruckes (2004). In Brunnermeier and Pedersen predatory behavior involves an opportunistic participant trading in the same direction as those in distress. In their model where there is no immediate price impact from predatory activity, the opportunistic trader trades at the same rate as the traders in distress. Our paper extends their work by endogenizing the price impact of trading, which also allows for price jumps during predatory incidents. Attari, Mello, and Ruckes (2004) use a two-period model to describe predatory trading behavior. The authors show that predators may even lend to others that are "financially fragile" because they can obtain higher profits by trading against them for a longer period of time. Our paper generalizes this type of model in a multi-period framework, with each period in a continuous-time setting. The main difference between our work and previous work by both Brunnermeier and Pedersen and Attari, Mello, and Ruckes is that we consider a model where the traders interact repeatedly over time. To our knowledge, this has not been addressed in the literature.

We use a pricing rule that accounts for both the effect of asset supply in the market and the rate of trading by participants on price formation. The effect of large trades on asset prices has been considered in empirical studies (Keim and Madhavan 1996; Kaul, Mehrotra, and Morck 2000; Holthausen, Leftwich, and Mayers 1990; Chan and Lakonishok 1995) and in theoretical studies (Bertsimas and Lo 1998; Fedyk 2001; DeMarzo and Uroševic 2000). A similar formulation of this pricing relationship has been previously derived by Vayanos (1998), as well as by Gennotte and Kyle (1991) who show that it arises from the equilibrium strategies between a market maker and an informed trader when the position of the noise traders follows a smoothed Brownian motion. Likewise, Pritsker (2004) obtains a similar relationship for the price impact of large trades when institutional investors transact in the market.

There is a large empirical literature that shows that cooperation affects price evolution in financial markets (Battalio, Ellul, and Jennings 2004; Cocco, Gomes, and Martins 2003; Berhardt, Dvoracek, Hughson, and Werner 2004; Desgranges and Foucault 2002; Reiss and Werner 2003; Ramadorai 2003; Hansch, Naik, and Viswanathan 1999; Massa and Simonov 2003). In addition to the papers by Cocco, Gomes, and Martins (2003) and Battalio, Ellul, and Jennings (2004) already cited, Massa and Simonov (2003) demonstrate evidence in the Italian Treasury bond market that "salient traders" who are either known to be smart, skeptical, or scared have a statistically significant effect on prices and on volatility in the market. This relates to our model in that we assume that traders are well known to each other and develop relationships. These relationships affect price formation in the market and they cooperate to provide liquidity to each other.

# 1 Trading and Predation

## 1 A Asset price model

The economy consists of two types of participants. The *strategic traders*, $i \in \{1, 2, ..., n\}$, are risk-neutral and maximize trading profits. These traders form a tight oligopoly over order flow in financial markets. These large traders are usually present in markets as proprietary trading desks, who trade both on their own account as well as for others. The strategic traders have inside information regarding transient liquidity needs within the market because they observe the order flow. Thus, they attempt to generate profits through their ability to forecast price moves, and to affect asset prices. The other players are the *long-term investors* who form the competitive fringe. The long-term investors usually trade in the interest of mutual funds or private clients and exhibit a less aggressive trading strategy. Long-term investors are more likely to take a "buy and hold strategy", limit the number of transactions that they undertake, and avoid taking over-leveraged positions. The long-term investors trade according to fundamentals. The primary difference between the two types of traders is that the long-term investors are not aware of transient liquidity needs in the economy.

There exist a risk-free asset and a risky asset, traded in continuous-time. The aggregate supply $S > 0$ of the risky asset at any time $t$ is divided between the strategic investors' holdings $X_t$ and the long-term investors' holdings $Z_t$ such that

$$S = X_t + Z_t. \tag{1}$$

The return on the risky asset is stochastic. The yield on the risk-free asset is zero.

The asset is traded at the price

$$P_t = U_t + \gamma X_t + \lambda Y_t, \tag{2}$$

where

$$dX_t = Y_t dt, \tag{3}$$

and $U_t$ is the stochastic process

$$dU_t = \sigma(t, U_t)dB_t, \tag{4}$$

with $B_t$ a one-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. The pricing equation is composed of three parts. $U_t$ represents the expected value of future dividends and is modelled as a martingale. The diffusion does not include a drift term. This is justified by the short-term nature of the events modelled. Most results described here can be derived with the inclusion of a drift term in the diffusion, but with considerable loss in clarity of exposition.[3]

The second term represents an inventory parameter in the economy and is present in the model by Brunnermeier and Pedersen. $X_t$ is the amount of the asset that the strategic traders hold at time $t$ and $\gamma$ is a market depth parameter. As $X_t$ increases, the supply available to the long-term investors decreases and the price at which they can access the asset increases.

The third term in the pricing formula measures the instantaneous price pressure that occurs through trading. This term is composed of a price impact parameter $\lambda$ and the rate of trade $Y_t$. The faster the traders sell, the lower a

[3]More precisely, the assumption is that the difference between the drift coefficient and the continuous-time discount factor is zero. For the multi-period game which we will later discuss, the assumption is that $T$ is relatively small, that is the distress and predation events develop over short periods of time, and the discounting over each period is therefore not significant. The period-to-period discount factor is then also close to one. Since each period is short, the multi-stage game will consist of many short periods, where the probabilities of a player being distressed in any given period are small, so that the period-to-period discount factor is significant to the problem.

price they will realize. This term models the transaction costs of trading and distinguishes our stage model from Brunnermeier and Pedersen (2004).

During a stage game (liquidity event), each trader seeks to maximize trading profits subject to initial and terminal holding constraints. Each trader solves the following dynamic program

$$\begin{aligned} \text{maximize} \quad & \textstyle\int_0^T -P_t Y_t \, dt \\ \text{subject to} \quad & X_0^i = x_0 \\ & X_T^i = x_T \end{aligned} \tag{5}$$

by choosing a trading rate $Y_t^i$, where the total net trading rate by the large traders is $Y_t = \sum_{i=1}^n Y_t^i$, and $dX_t^i = Y_t^i dt$. In the equilibrium, each trader chooses an action (trade) that is the best response to the other traders' actions The trading rate is a stochastic process, since it can depend on previous prices. It must, however, be $\mathcal{F}_t$-adapted, that is, decisions cannot depend on future prices.

In the following subsections, we describe the equilibrium optimal trading policies under different scenarios, and evaluate expected values and surplus loss effects that occur when there exist predators in the market and when there exists selling competition between distressed traders.

## 1 B Single large trader

The following result explores the optimal trading rule when a trader who needs to trade has monopoly power. That is, the trader buys or sells in the absence of other strategic traders. The optimal trading policy for a single large trader

($n = 1,\ Y_t = Y_t^1$) is to trade at a constant rate. Without loss of generality, we assume that the initial position in the asset is zero.

**Result 1** *(One Trader) Consider a trader with initial position* $X_0 = 0$ *and target* $X_T = \Delta x$ *in an asset with price process*

$$\begin{aligned} P_t &= U_t + \gamma X_t + \lambda Y_t \\ dX_t &= Y_t\, dt \\ dU_t &= \sigma(t, U_t)\, dB_t, \end{aligned} \tag{6}$$

*where* $B_t$ *is a Brownian motion on* $(\Omega, \mathcal{F}, P)$*. The trader maximizes*

$$\mathbf{E} \int_0^T -P_t Y_t dt, \tag{7}$$

*over the* $\mathcal{F}_t$*-adapted policies for the trading rate* $Y_t$*. Then, the optimal policy is*

$$Y_t = \frac{1}{T-t}\,(\Delta x - X_t) = \frac{\Delta x}{T}. \tag{8}$$

*The expected value for the single large trader is*

$$V_1 = -u\,\Delta x - \left(\frac{\gamma}{2} + \frac{\lambda}{T}\right) \Delta x^2. \tag{9}$$

**Proof.** Proof by verification. Consider the value function

$$\phi_t(x, u) = -u(\Delta x - x) - \frac{\gamma}{2}(\Delta x^2 - x^2) - \frac{\lambda}{T-t}(\Delta x - x)^2. \tag{10}$$

To verify the result, apply to $\phi$ the generator of the process with policy $y$,

$$\begin{aligned} L^y \phi &= \frac{\partial \phi}{\partial x}\, y + \frac{\partial^2 \phi}{\partial u^2}\, \sigma(t, u) + \frac{\partial \phi}{\partial t} && (11) \\ &= \left(u + \gamma x + 2\frac{\lambda}{T-t}(\Delta x - x)\right) y - \frac{\lambda}{(T-t)^2}(\Delta x - x)^2. && (12) \end{aligned}$$

The Hamilton-Jacobi-Bellman (HJB) equation is satisfied since

$$F + L^y \phi \;=\; -\lambda \left( y - \frac{1}{T-t}(\Delta x - x) \right)^2 \tag{13}$$

is maximized to zero with policy (8), where $F = -(u+\gamma x+\lambda y)y$ is the integrand of (7). The value $V_1$ is derived directly from the value function. ■

The key finding of this result is that it is optimal for the distressed seller to smooth his order flow and sell at a constant rate. Further, the expected value to a distressed trader when no other strategic traders are present is the best value that a distressed trader can derive. When other players trade strategically at the same time in competition or against a distressed trader, the value that the trader can derive is strictly lower that $V_1$. We will formalize this assertion in the next subsection.

## 1 C Multiple traders, general solution

We now describe the general structure of the equilibrium trading policies for the game with multiple traders. This formulation will serve a basis for deriving the equilibrium strategies when several distressed traders are present without opportunism and when there are both opportunistic and distressed traders present in the economy. We begin by considering the deterministic case where $U_t = u$, for some constant $u$ and all $t$. We will then show that the solution to this problem also solves the stochastic case.

**Result 2** *(Multiple Traders with Deterministic Price) Consider $n$ traders, each* *with position $X_t^i$ and trading at rate $Y_t^i$. The asset price is*

$$P_t = u + \gamma \sum_{j=1}^{n} X_t^j + \lambda \sum_{j=1}^{n} Y_t^j, \tag{14}$$

*where $u \in \mathbf{R}$ is a constant. The traders are all mutually informed of their initial positions $x_{0i}$ and trading targets $x_{Ti}$. Trader $i$ chooses $Y_t^i$ to solve the optimization problem*

$$\begin{aligned} &\textit{maximize} && \textstyle\int_0^T -P_t Y_t^i \, dt \\ &\textit{subject to} && dX_t^j = Y_t^j dt, \; j = 1, \ldots, n, \\ & && X_0^j = x_{0j}, \; X_T^j = x_{Tj}, \; j = 1, \ldots, n, \end{aligned} \tag{15}$$

*assuming that the other traders also trade optimally, that is that $Y_t^j, j \neq i$, are solutions to equivalent problems. Then, the equilibrium optimal policies are of the form*

$$Y_t^i = a \, e^{-\frac{n-1}{n+1} \frac{\gamma}{\lambda} t} + \sum_{k=0}^{n-2} b_{ik} t^k e^{\frac{\gamma}{\lambda} t}, \tag{16}$$

*with $b$ such that $\sum_{i=1}^n b_{ik} = 0$ for each $k$.*[4]

**Proof.** Introducing the multiplier function $Z_t^i$, necessary optimality conditions are

$$\begin{aligned} & u + \gamma \textstyle\sum_{j=1}^n X_t^j + \lambda \sum_{j=1}^n Y_t^j + \lambda \, Y_t^i + Z_t^i = 0 \\ & dZ_t^i = -\gamma \, Y_t^i dt. \end{aligned} \tag{17}$$

[4] These policies constitute an equilibrium over open-loop policies. The open-loop policy is consistent with trading in the markets of interest. Each player is informed about the number of traders present, and about the other players' trading targets. However, once a trading strategy is initiated, information about each player's current instantaneous trading rate is generally not immediately available. We have also analyzed closed-loop policies, and even though the solution in this case cannot be expressed in closed-form, it is qualitatively similar.

Differentiating the first equation with respect to $t$, and substituting the second,

$$\gamma \sum_{j=1}^{n} Y_t^j dt + \lambda \sum_{j=1}^{n} dY_t^j + \lambda dY_t^i - \gamma Y_t^i dt = 0. \tag{18}$$

The $n$ such equations for each $i$ can be written together as

$$\lambda(I + \mathbf{1})dY = \gamma(I - \mathbf{1})Y dt, \tag{19}$$

where $I$ is the $n \times n$ identity matrix, and $\mathbf{1}$ is the $n \times n$ matrix with all elements equal to one. From the formula for the inverse of the rank-one update of a matrix, the inverse of the left-hand-side matrix is

$$(I + \mathbf{1})^{-1} = I - \frac{1}{n+1}\mathbf{1}, \tag{20}$$

which we use to write the linear dynamic system in the form

$$dY = \frac{\gamma}{\lambda} A Y dt, \quad A = I - \frac{2}{n+1}\mathbf{1}. \tag{21}$$

Denote by $\bar{1}$ the $n$-vector with all entries equal to one. Since

$$A\bar{1} = \bar{1} - \frac{2}{n+1} n\bar{1} = -\frac{n-1}{n+1}\,\bar{1}, \tag{22}$$

the vector of ones is an eigenvector of the matrix $A$, with associated eigenvalue $-\frac{n-1}{n+1}$. Likewise, vectors in the null-space of $\bar{1}$ are eigenvectors of $A$, with eigenvalue 1: for $v$ orthogonal to the vector of ones, that is satisfying $\bar{1}^T v = 0$,

$$Av = v - \frac{2}{n+1}\mathbf{1}v = v. \tag{23}$$

The dimension of this sub-space, and multiplicity of the eigenvalue 1, is $n - 1$. Solutions to the system of linear differential equations are therefore as in (16), and equilibrium optimal policies must be of this form. ■

The trading dynamic derived in Result 2 has a unique solution for the cases when the number of traders, $n = 2$, and when all of the traders are distressed (not acting as predators). We will illustrate these cases in Section 1 D and Section 1 E. For $n > 2$ and the presence of predators, the equilibrium optimal policies are in general not unique, as $b$ can lie anywhere in an affine sub-space of dimension $n - 2$. This non-uniqueness arises from our pricing equation as $Y_t = \sum_{i=1}^{n} Y_t^i$. Each trader $i$ chooses $Y_t^i$ based on $\sum_{j \neq i} Y_t^j$. Therefore, any combination of $Y_t^j$ for $j \neq i$ will lead to the same trading dynamic. Hence, we observe multiple equilibrium when multiple predators and distressed traders coexist. This non-uniqueness does not preclude surplus analysis, which is described below. Before deriving expected value results for multiple traders, we show that, on account of the Martingale property of $U_t$, the solution to the deterministic case also solves the stochastic case.

**Result 3** *(Multiple Traders with Stochastic Price) Consider the $n$ trader problem as in Result 2, but with the asset price following the stochastic process*

$$\begin{aligned} P_t &= U_t + \gamma \sum_{j=1}^{n} X_t^j + \lambda \sum_{j=1}^{n} Y_t^j \\ dU_t &= \sigma(t, U_t)\, dB_t, \end{aligned} \tag{24}$$

*where $B_t$ is a Brownian motion on $(\Omega, \mathcal{F}, P)$, and trader $i$ maximizing*

$$\mathbf{E} \int_0^T -P_t Y_t^i dt, \tag{25}$$

*over the $\mathcal{F}_t$-adapted policies for the trading rate $Y_t^i$. Then, the equilibrium optimal policies are as in (16).*

**Proof.** Since the objectives for the deterministic case are linear in $u$, and the corresponding optimal policies do not depend on $u$, the value functions are linear in $u$. Using these value functions for the stochastic case, the $\partial^2 \phi_i / \partial u^2$ terms in the HJB equations are zero, and all other terms are as in the HJB equations for the deterministic case. That is, if given value functions and optimal policies satisfy the HJB equations for the deterministic case, they also satisfy the HJB equations for the stochastic case. ■

As noted above, if $n > 2$ and some of the traders are predators, the equilibrium optimal policies are in general not unique, as $b$ can lie anywhere in an affine sub-space of dimension $n - 2$. The non-uniqueness of the equilibrium is significant because different equilibria correspond to different expected values for each trader. However, the coefficient $a$ is always uniquely determined. Since the total holdings by the large strategic traders only depends on the constant $a$, the total surplus is the same for all solutions.

The following result generalizes the surplus effects of our model, which we will apply to the two-trader stage game in Section 1 E. We define $V_n$ as the total surplus in the market when $n$ traders play this game and we define $\Delta V_n$ as the change in surplus that occurs compared to the expected value derived in Section 1 B for one distressed trader. We find that the loss in surplus is increasing with the number of traders.

**Result 4** *(Expected Total Surplus and Loss for Multiple Traders) The total* *surplus for $n$ traders with a combined trading target $\Delta x$ is*

$$V_n \;=\; -\,u\,\Delta x - \frac{\gamma}{2}\left(1 + \frac{n-1}{n+1}\cdot\frac{1+e^{-\frac{n-1}{n+1}\Gamma}}{1-e^{-\frac{n-1}{n+1}\Gamma}}\right)\Delta x^2. \tag{26}$$

*where $\Gamma = \frac{\gamma}{\lambda}T$. For $\lambda$ small ($\Gamma \to \infty$),*

$$V_n \;\to\; -\,u\,\Delta x - \frac{n}{n+1}\gamma\,\Delta x^2. \tag{27}$$

*The expected loss in total surplus from competition is*

$$\Delta V_n \;=\; V_1 - V_n \;=\; \gamma\left(\frac{1}{2}\cdot\frac{n-1}{n+1}\cdot\frac{1+e^{-\frac{n-1}{n+1}\Gamma}}{1-e^{-\frac{n-1}{n+1}\Gamma}} - \frac{1}{\Gamma}\right)\Delta x^2. \tag{28}$$

*$\Delta V_n$ is monotonic increasing in $T$, $\gamma$, and $n$, and monotonic decreasing in $\lambda$. For $\lambda$ small ($\Gamma \to \infty$),*

$$\Delta V_n \;\to\; \frac{1}{2}\cdot\frac{n-1}{n+1}\gamma\,\Delta x^2, \tag{29}$$

*and for $\lambda$ large ($\Gamma \to 0$),*

$$\Delta V_n \;\to\; 0. \tag{30}$$

**Proof.** By integration of $-P_t\sum_{i=1}^{n} Y_t^i \;=\; -P_t\,n\,a\,e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}t}$ over $t \in [0,T]$, followed by algebraic simplification. The monotonicities are verified by differentiation, and the limits follow by applying l'Hôpital's rule as needed. ■

## 1 D Multiple distressed traders, symmetry, absence of predators

We now use the general solution from Section 1 C to derive the trading equilibrium for multiple traders who are distressed in the absence of any predatory behavior. The traders "race" to market in a decreasing exponential fashion. As the number of traders increases, the trading intensity increases, approaching a finite trading intensity in the limit.

**Result 5** *(Multiple Identical Traders) Consider $n$ traders, with identical trading targets $\Delta x_i = \Delta x/n$. The unique symmetric equilibrium strategy is*

$$Y_t^i = a\, e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}t},\ i = 1, \ldots, n. \tag{31}$$

where

$$a = \frac{n-1}{n+1}\frac{\gamma}{\lambda}\left(1 - e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}T}\right)^{-1}\frac{\Delta x}{n}. \tag{32}$$

**Proof.** This is a direct consequence of Result 3. ■

If the market is deeper (small $\gamma$), trading will occur comparatively later. If the short-term price impact of trading is smaller (small $\lambda$), trading will occur comparatively earlier. Figure 1 plots the trading policies for multiple symmetric traders with $\Delta x = 1$, $T = 1$, and $\frac{\gamma}{\lambda} = 10$. As expected, for $n = 1$ we recover the constant selling rate of Section 1 B. If there are more traders, everybody will trade earlier. The rate of trade goes to $e^{-\frac{\gamma}{\lambda}t}$ as $n \to \infty$. That is, there is an upper bound on how fast traders will sell their position, regardless of how many traders are in the race.

Figure 2 plots the corresponding price process (for $U_t = u$ constant). For a single trader, the price changes linearly over the trading period. For a large number of traders, the price function over $t \in [0, T]$ goes to a constant value. That is, the information regarding the trader's target position in the asset becomes fully incorporated in the asset price quickly.

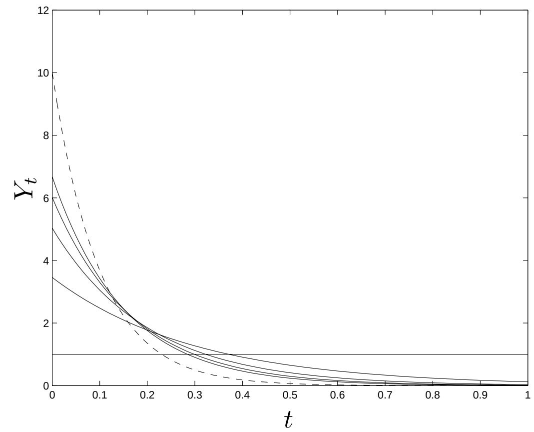


Figure 1: Competition among traders leads to a 'race to trade'. Trading rate for multiple traders with identical targets (solid for $n = 1, 2, 3, 4, 5$, dotted for $n = \infty$). Parameters for example are $\Delta x = 1,\ T = 1,\ \gamma/\lambda = 10$.

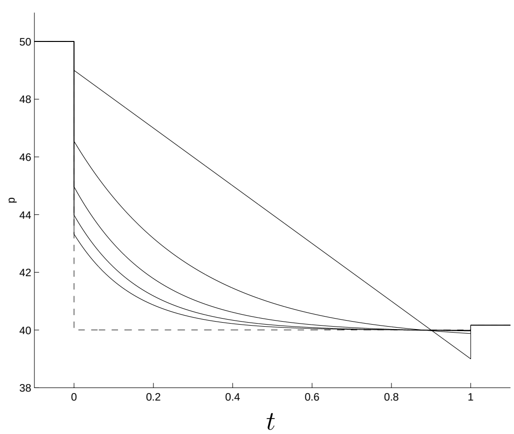


Figure 2: Price for multiple traders with identical targets (solid for $n = 1, 2, 3, 4, 5$, dotted for $n = \infty$). Parameters for example are $\Delta x = -1,\ T = 1,\ \gamma/\lambda = 10$.

## 1 E Stage game with two traders

We now set up and analyze the two-player predatory stage game, which will form the basis for the infinitely-repeated game in Section 2. Consider one distressed and one opportunistic trader, and a game which proceeds as follows. A liquidity event occurs at time $t = 0$, whereby one of the two strategic traders is required buy or sell a large block of the asset in a short time horizon (say, by the end of the trading day). Forced liquidation usually arises because of the need to offset another cash-constrained position such as an over-leveraged position, or it occurs as a result of a risk management maneuver. The second trader is informed of the liquidity event, and of the trading requirement of the distressed trader. The opportunistic trader returns to his original position in the asset by the end of the trading period, but will trade strategically to exploit the price impact of the distressed trader's selling. Each trader chooses a trading schedule over the period to maximize his own expected value, assuming the other trader will do likewise.

**Result 6** *(One Distressed Trader and One Predatory Trader) Consider two traders, under the model and assumptions of Result 3. A distressed trader is required to change position in the asset by $\Delta x$ in time $T$. A predatory trader is required to return to the initial position in the asset by time $T$. The unique equilibrium trading policies are*

$$\begin{aligned} Y_t^d &= a\,e^{-\frac{1}{3}\frac{\gamma}{\lambda}t} + b\,e^{\frac{\gamma}{\lambda}t} \\ Y_t^p &= a\,e^{-\frac{1}{3}\frac{\gamma}{\lambda}t} - b\,e^{\frac{\gamma}{\lambda}t}, \end{aligned} \tag{33}$$

*where*

$$\begin{aligned} a &= \tfrac{1}{6}\tfrac{\gamma}{\lambda}\left(1 - e^{-\frac{1}{3}\frac{\gamma}{\lambda}T}\right)^{-1}\Delta x \\ b &= \tfrac{1}{2}\tfrac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T} - 1\right)^{-1}\Delta x. \end{aligned} \tag{34}$$

**Proof.** This follows from Result 3, and from the constraints on total trading

$$\begin{aligned} -3a\tfrac{\lambda}{\gamma}\left(e^{-\frac{1}{3}\frac{\gamma}{\lambda}T} - 1\right) + b\tfrac{\lambda}{\gamma}\left(e^{\frac{\gamma}{\lambda}T} - 1\right) &= \Delta x \\ -3a\tfrac{\lambda}{\gamma}\left(e^{-\frac{1}{3}\frac{\gamma}{\lambda}T} - 1\right) - b\tfrac{\lambda}{\gamma}\left(e^{\frac{\gamma}{\lambda}T} - 1\right) &= 0. \end{aligned} \tag{35}$$

■

For $\Delta x_d > 0$, the solution is such that $a > 0$ and $b > 0$. The shape of the trading strategy depends on the parameters of the market. Figure 3 gives an example, with $\Delta x_d = 1$, $T = 1$, and $\frac{\gamma}{\lambda} = 10$. The strategy involves the opportunistic trader initially racing the distressed trader to the market in an exponential fashion, and then fading the distressed trader towards the end of the period, also exponentially. If the first trader needs to sell, that is $\Delta x_d < 0$, the predatory trader sells short at beginning and buys back in later periods to cover his position. If the distressed trader is required to buy a block of the asset, the opposite strategy by the predator ensues. In general, we see that the presence of the predator will lead the distressed trader to increase his trading volume at the beginning and at the end of the trading period. This leads to a U-shaped trading volume over the period, a pattern observed in most markets.[5]

[5] A variation on this model would be to allow the opportunistic trader to trade over a longer time period than the distressed trader. The solution for such a model is similar, with racing and fading occurring while the distressed seller is trading. However, the predatory trader will now choose what position to have by the end of the distressed seller's deadline. This choice is made by maximizing the expected value from trading over the distressed seller's period,

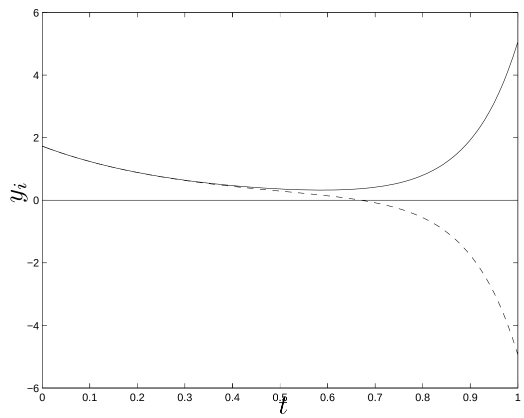


Figure 3: One trader with position target (solid) and one 'opportunistic' trader (dotted). Parameters for example are $\Delta x_d = 1, T = 1, \gamma/\lambda = 10$.

Now we apply Result 4 to the two-trader case and derive a surplus result that we will use in Section 2. We define $V_2$ as the total expected value for the large traders when two traders play this game, and we define $V_d$ and $V_p$ as the expected values to the distressed and opportunistic traders. Likewise, we define $\Delta V_2$ as the change in surplus that occurs compared to the expected value, $V_1$, derived in Section 1 B for a single distressed trader. In the following result, we show that $\Delta V_2$ is strictly positive.

**Result 7** *(Expected Total Surplus and Loss for Two Traders) The total surplus for the distressed trader and the predatory trader is*

$$V_2 \;=\; V_d + V_p \;=\; -\,u\,\Delta x - \frac{\gamma}{3} \cdot \frac{2 - e^{-\frac{1}{3}\Gamma}}{1 - e^{-\frac{1}{3}\Gamma}}\,\Delta x^2. \tag{36}$$

plus the expected value from selling the position at the end of that period at a constant rate over the additional time. For ease of exposition, we restrict our discussion to a common time period for all traders.

*where* $\Gamma = \frac{\gamma}{\lambda}T$*. For* $\lambda$ *small* ($\Gamma \to \infty$)*,*

$$V_2 \;\to\; -u\,\Delta x - \frac{2}{3}\gamma\,\Delta x^2. \tag{37}$$

*The expected loss in total surplus from predation is*

$$\Delta V_2 \;=\; V_1 - V_2 \;=\; \gamma\left(\frac{1}{6}\cdot\frac{1+e^{-\frac{1}{3}\Gamma}}{1-e^{-\frac{1}{3}\Gamma}} - \frac{1}{\Gamma}\right)\Delta x^2. \tag{38}$$

$\Delta V_2$ *is positive, monotonic increasing in* $T$ *and in* $\gamma$*, and monotonic decreasing in* $\lambda$*. For* $\lambda$ *small* ($\Gamma \to \infty$)*,*

$$\Delta V_2 \;\to\; \frac{1}{6}\gamma\,\Delta x^2, \tag{39}$$

*and for* $\lambda$ *large* ($\Gamma \to 0$)*,*

$$\Delta V_2 \;\to\; 0. \tag{40}$$

**Proof.** By equation (33), $Y = Y_t^d + Y_t^p = 2ae^{-\frac{1}{3}\frac{\gamma}{\lambda}t}$. By integration of $-P_tY$ over $t \in [0,T]$, followed by algebraic simplification, the results in equations (36) and (38) are derived. The monotonicities are verified by differentiation of equation (38). The limits follow by applying l'Hôpital's rule as necessary. ■

The limit for $\lambda$ small provides some economic intuition regarding the magnitude of the effects. If we examine $\Delta V_2$ as $\lambda$ becomes small, we see that the total change in surplus goes to $\frac{1}{6}\gamma\,\Delta x^2$. Given our result from Section 1 B, we can see that the predator gains $\frac{1}{6}\gamma\,\Delta x^2$ and the distressed seller loses $\frac{1}{3}\gamma\,\Delta x^2$. That is, in the limit, the predator gains half of what the distressed trader loses. To generalize this finding, we derive the following result.

**Result 8** *(Expected Value for Two Traders) The expected values for the dis-*

*tressed trader and for the predatory trader are*

$$\begin{aligned} V_d &= -u\,\Delta x - \frac{\gamma}{6}\,\frac{5e^{\Gamma}+e^{\frac{2}{3}\Gamma}+e^{\frac{1}{3}\Gamma}-1}{e^{\Gamma}-1}\,\Delta x^2, \\ V_p &= \frac{\gamma}{6}\cdot\frac{e^{\frac{2}{3}\Gamma}-1}{e^{\frac{2}{3}\Gamma}+e^{\frac{1}{3}\Gamma}+1}\,\Delta x^2, \end{aligned} \tag{41}$$

*where* $\Gamma = \gamma/\lambda$*. For* $\lambda$ *small (*$\Gamma \to \infty$*),*

$$\begin{aligned} V_d &\to -u\,\Delta x - \frac{5}{6}\gamma\,\Delta x^2, \\ V_p &\to \frac{1}{6}\gamma\,\Delta x^2. \end{aligned} \tag{42}$$

**Proof.** We define $V_- = V_d - V_p$ and $Y_- = Y_t^d - Y_t^p = 2be^{\frac{\gamma}{\lambda}t}$. Integrate $-P_tY_-$ over $t \in [0,T]$ and simplify to obtain

$$V_- = V_d - V_p = -u\,\Delta x - \gamma\,\frac{e^{\Gamma}}{e^{\Gamma}-1}\,\Delta x^2. \tag{43}$$

$V_d$ and $V_p$ are obtained by simplification of $(V_2+V_-)/2$ and $(V_2-V_-)/2$. ■

# 2 Cooperation and Liquidity

## 2 A Repeated Game

The repeated game is based on the case in which there are two strategic traders, as well as a large number of long-term investors. Each player faces a common discount factor $\delta$, and the common asset price determinants $u$, $\gamma$, and $\lambda$. At the beginning of each stage, nature moves first, assigns a type to each of the traders and both traders know each other's type in each round. In each round, each trader, with probability $p_i$ $i = 1, 2$, must liquidate a large position of size $\Delta x$, and may act as a predator with probability $1 - p_i$. The magnitude of the shock is constant. An alternative approach, which we take in Section 3, is to model $\Delta x$ as a random variable, and compute the value of the supergame by expectation over future liquidity events.

In each time period one of the following events occurs: neither of the two players is distressed, with probability $p_{00}$; the second player is distressed but the first is not, with probability $p_{01}$; the first player is distressed but the second is not, with probability $p_{10}$; both players are distressed, with probability $p_{11}$. The four probabilities add to one. If the probability of each of the two players being distressed is independent of the other, we have $p_{00} = (1 - p_1)(1 - p_2)$, $p_{01} = (1 - p_1)p_2$, $p_{10} = p_1(1 - p_2)$, and $p_{11} = p_1 p_2$. Cooperation is possible when either there exists one predator and one distressed trader (with probability $p_{10} + p_{01}$), or when both players are distressed (with probability $p_{11}$). If only one of the players is distressed and needs to liquidate a position, cooperation involves the other refraining from engaging in predatory trading. If both traders are distressed, cooperation involves both traders selling at a constant rate and refraining from racing each other to the market for their own gain.

Cooperation provides the players with the ability to quickly sell large blocks of shares, for the price that would be obtained by selling them progressively over time. That is, while cooperation is ongoing, the distressed trader is allowed to "ride down" the demand curve, rather than having the information regarding the trading target quickly incorporated into the asset price, ahead of most of his trading. In this sense, that large blocks of shares can be moved for a better price, the market will appear more liquid. It will also avoid the volatility and potential instability from the large trading volume peaks associated with the racing and fading.

The punishment strategy considered is a trigger strategy in the spirit of Dutta and Madhavan (1997) and Rotemberg and Saloner (1986). [6] We investigate the effects of cooperation on liquidity by contrasting the most collusive equilibrium with the trigger strategy that uses as punishment the inferior stage game equilibrium derived in Section 1 E.

**Result 9** *(Repeated Game with Two Symmetric Traders) Define the expected values as in Section 1. The discount factor required to support collusion is*

$$\delta \geq \frac{V_p}{p_{10}[V_1 - V_d] + \frac{1}{2}p_{11}[V_1(2\Delta x) - V_2(2\Delta x)] + (1 - p_{01})V_p}. \tag{44}$$

*Substituting for the values previously derived this can be expressed as*

$$\begin{aligned} \delta \quad \geq \quad & \Bigg\{ 1 - p_{01} + p_{10}\left[\frac{2e^{\frac{2}{3}\frac{\gamma}{\lambda}T} - e^{\frac{1}{3}\frac{\gamma}{\lambda}T} + 2}{(e^{\frac{1}{3}\frac{\gamma}{\lambda}T} - 1)^2} - \frac{2(e^{\frac{2}{3}\frac{\gamma}{\lambda}T} + e^{\frac{1}{3}\frac{\gamma}{\lambda}T} + 1)}{(e^{\frac{1}{3}\frac{\gamma}{\lambda}T} - 1)(e^{\frac{1}{3}\frac{\gamma}{\lambda}T} + 1)\frac{1}{3}\frac{\gamma}{\lambda}T}\right] \\ & + p_{11}\left[\frac{2(e^{\frac{2}{3}\frac{\gamma}{\lambda}T} + e^{\frac{1}{3}\frac{\gamma}{\lambda}T} + 1)}{(e^{\frac{1}{3}\frac{\gamma}{\lambda}T} - 1)^2} - \frac{4(e^{\frac{2}{3}\frac{\gamma}{\lambda}T} + e^{\frac{1}{3}\frac{\gamma}{\lambda}T} + 1)}{(e^{\frac{1}{3}\frac{\gamma}{\lambda}T} - 1)(e^{\frac{1}{3}\frac{\gamma}{\lambda}T} + 1)\frac{1}{3}\frac{\gamma}{\lambda}T}\right] \Bigg\}^{-1}. \end{aligned} \tag{45}$$

*For $\lambda$ small ($\frac{\gamma}{\lambda}T \to \infty$),*

$$\delta \geq \frac{1}{2p_{10} - p_{01} + 2p_{11} + 1}. \tag{46}$$

*For $\lambda$ large ($\frac{\gamma}{\lambda}T \to 0$)*

$$\delta \geq 0. \tag{47}$$

[6]Other alternative punishment schemes include those in which cooperation is abandoned for a fixed number of periods and the more complicated punishment schemes of Abreu (1988). Using such alternative punishment strategies does not lead to different economic results.

**Proof.** Define $\chi = e^{\frac{1}{3}\frac{\gamma}{\lambda}T}$. In the non-cooperative stage game, with probability $p_{00}$ each player earns zero. With probabilities $p_{10}$ or $p_{01}$, the predator earns $V_p$ and the distressed seller earns $V_d$ such that

$$V_p = \frac{\gamma}{6}\frac{\chi^2 - 1}{\chi^2 + \chi + 1}\Delta x^2 \tag{48}$$

and

$$V_d = -u\Delta x - \frac{\gamma}{6}\frac{5\chi^3 + \chi^2 + \chi - 1}{\chi^3 - 1}\Delta x^2. \tag{49}$$

With probability $p_{11}$ both traders are distressed and earn

$$V_r = \frac{1}{2}V_2(2\Delta x) = -u\Delta x - \frac{2\gamma}{3}\frac{2\chi - 1}{\chi - 1}\Delta x^2. \tag{50}$$

In the cooperative stage game when either player is a predator, they earn zero. When a player is distressed and faces a potential predator, they earn

$$V_1(\Delta x) = -u\Delta x - \frac{\gamma}{2}(1 + \frac{2}{3}\frac{1}{\log \chi})\Delta x^2. \tag{51}$$

and when both traders are in distress, they each earn

$$\frac{1}{2}V_1(2\Delta x) = -u\Delta x - \gamma(1 + \frac{2}{3}\frac{1}{\log \chi})\Delta x^2. \tag{52}$$

To calculate the $\delta$ necessary to support cooperation, the gains to collusion must exceed those of one-time deviation and infinite non-cooperation or

$$\frac{\delta}{1-\delta}[p_{10}V_1(\Delta x) + \frac{1}{2}p_{11}V_1(2\Delta x)] \geq V_p + \frac{\delta}{1-\delta}[p_{01}V_p + p_{10}V_d + \frac{1}{2}p_{11}V_2(2\Delta x)] \tag{53}$$

Equation (44) follows from algebraic manipulation. Equation (45) is obtained from substitution and further algebraic manipulation. The limits are found by using l'Hôpital's rule as needed. ■

This result implies that cooperation can be sustained as long as both traders are sufficiently patient. The trader's probability of being in distress affects the potential for collusion. If $\lambda$ is large, then it is easy to support cooperation. As the price impact of trading decreases, the relative probability of distress becomes more important. From equation (46), it can be shown that if

$$p_{01} \geq 2(p_{10} - p_{11}) \tag{54}$$

then there is no $\delta$ that supports cooperation. An important consequence of this fact is that if the probability of distress is linked to market share, it may benefit a large trader to allow a smaller trader to grow in size so that a Pareto superior outcome for the strategic traders can be achieved or maintained. We will see in the next section that this relationship becomes important when there is competition for business with external players.

## 2 B Trading with Outsiders

Now, we consider the relationship between a member of the cooperating oligopoly and an outsider who seeks to trade a large block of the asset. Because this represents a one-shot stage game, the members of the oligopoly have two alternatives. They may initiate a predatory strategy, race and fade the outsider to the market, and earn a profit by manipulating the price. Alternatively, the oligopoly members may exact appropriate rents from the outsider for the use of their services as members of the cartel (these rents may arise in the form of a bid-ask spread). The fact that there exists a cooperative outcome in this market between the insiders provides a means by which a relatively stable, albeit widened, bid-ask spread may exist, and we do not necessarily observe price volatility when a non-member needs liquidity.

We define $V^*$ as the difference between $V_1$ and $V_d(\Delta x)$. Since the outsider would be indifferent between receiving $V_d(\Delta x)$ and paying $V^*$ in order to receive $V_1$ when trading, the total surplus available is $V^*$. To determine the division of the surplus between the insiders and the external player, we use an generalized Nash bargaining solution. We assume that the insiders receive fraction $\tau$ of the the surplus and the external player receives fraction $1 - \tau$.

In the next result, we demonstrate the requirements for collusion in the presence of an external party who wishes to utilize the services of the cartel. We define $\pi_1$ and $\pi_2$ as the probability that the external trader is a customer of each of the traders. In this way $\pi_i$ represents a strategic trader's market share. Further, we define $\frac{1}{2}V_p$ as the value realized by a predator when two predators trade against one who is distressed. This result describes the case where, at any given time, only one trader is in the 'distressed' position, but can be extended to the general case in which members of the cartel may be simultaneously distressed.

**Result 10** *(Equilibrium with Two Insiders and an Outsider) Define $q$ as the probability that an external player needs to trade the asset in each period and define $\pi = min[\pi_1, \pi_2]$. For given levels of $V^*$, $\tau$, and $\pi$, the $\delta$ required to support collusion is*

$$\delta \geq \frac{V_p - q\pi\tau V^*}{V_p - \frac{1}{2}qV_p}. \tag{55}$$

*In order to maintain cooperation, each player's 'market share' must satisfy*

$$\pi_i \geq \frac{(1-\delta)V_p - \frac{1}{2}q\delta V_p}{q\tau V^*}. \tag{56}$$

*Since* $\delta \in [0,1]$*, then it must be that*

$$\pi \geq \frac{1}{2}\frac{V_p}{\tau V^*}. \tag{57}$$

**Proof.** Proceed as in Result 9. ■

From equation (56), we see that the market share of external business that each trader has impacts the ability to maintain cooperation. The model also suggests that deviations in the bid-ask spread may be observed in practice without resulting in price wars. The trader with a larger market share may be willing to allow a smaller player a better competitive position, so that their market share can increase. As the market shares converge, cooperation can be sustained.

Note that we have assumed so far that the outsider is not allowed to split the order flow while, in practice, order flow splitting is common. This is consistent with the results above. By, in effect, evening out the market share, the outside player reduces the chance that cooperation will break down (which would lead to a loss for the outside player).

# 3 Episodic Illiquidity and Contagion

## 3 A Shocks of Random Magnitude and Episodic Illiquidity

In Section 2, we evaluated the requirements for cooperation given that $\Delta x$ is a fixed amount of the asset. In that formulation, cooperation is sustained and the traders never deviate. To characterize episodic illiquidity, $\Delta x$ is better modeled as a random variable. In the event of a large $\Delta x$ it is more profitable for the traders to deviate for a one-time gain. However, instead of initiating the grim-trigger strategy outlined in Section 2, there are more profitable strategies available to the cartel. One approach is along the lines of Rotemberg and Saloner (1986).

The large traders implicitly agree to restrain from predating when the magnitude of the shock is below some threshold $\widetilde{\Delta x}$ and, conversely, not to punish other players in future periods for predating when the shock is above that threshold. That is, when a player has trading requirement that exceeds $\widetilde{\Delta x}$, the other player will predate, but cooperation is resumed in subsequent periods. This equilibrium behavior results in episodically increased volatility[7].

Another way to describe this equilibrium is that each trader agrees to restrain from predating on the other, but only as long as they behave 'responsibly' in their risk management. This creates a natural restriction on the exposure that each trader can take without a substantial increase in the risk of their portfolio.

The value of $\widetilde{\Delta x}$ which is optimal for the cartel (in the sense of leading to the highest expected value for its members) can be computed for any distribution of the trading requirement for each player. In general, $\widetilde{\Delta x}$ can only be characterized implicitly.

[7] Episodic illiquidity also occurs during extreme financial distress. During extreme distress, a member of the oligopoly becomes a finite concern. Because the horizon of this game is finite, the players work out their strategy profiles by backwards induction and cooperation disappears.

**Result 11** *(Shocks of Random Magnitude) Consider that the trading requirements for each of two players are random shock magnitudes* $\Delta x$ *that are distributed i.i.d. according to the density* $f(y)$*, which we assume to be*

*(i) symmetric,* $f(y) = f(-y)$*,*

*(ii) with unbounded support,* $f(y) > 0, \forall y$*,*

*(iii) and with finite variance,* $\int_{-\infty}^{\infty} y^2 f(y) dy < \infty$*.*

*A strategy with episodic predation with threshold* $\widetilde{\Delta x}$ *is feasible with any* $\widetilde{\Delta x}$ *that satisfies*

$$2C \int_0^{\widetilde{\Delta x}} y^2 f(y) dy \geq K \widetilde{\Delta x}^2. \tag{58}$$

*The supremum of* $\widetilde{\Delta x}$ *such that the inequality is satisfied exists, and we designate it by* $\overline{\Delta x}$*. The following strategy profile constitutes a sub-game perfect Nash equilibrium. At time* $t = 0$*, we predate if* $\Delta x > \widetilde{\Delta x}$*, and otherwise cooperate. At time* $t \neq 0$*,*

1. *If the history of play* $h^{t-1}$ *is such that for every period in which* $\Delta x < \widetilde{\Delta x}$ *there was no predation, then*

   *(a) If* $\Delta x > \widetilde{\Delta x}$*, predate this period.*

   *(b) If* $\Delta x < \widetilde{\Delta x}$*, cooperate.*

2. *If* $h^{t-1}$ *is such that for* $\Delta x < \widetilde{\Delta x}$*, there was predation, then predate.*

*The constants above are*

$$C = \frac{\delta}{1-\delta} \left[ p_{10} \left( K_d - \left( \frac{\gamma}{2} - \frac{\lambda}{T} \right) \right) + 2p_{11} \left( K_2 - \left( \frac{\gamma}{2} - \frac{\lambda}{T} \right) \right) - p_{01} K \right]$$

$$
\begin{aligned}
K &= \frac{\gamma}{6}\frac{e^{\frac{2}{3}\Gamma}-1}{e^{\frac{2}{3}\Gamma}+e^{\frac{1}{3}\Gamma}+1} \\
K_d &= \frac{\gamma}{6}\frac{5e^{\Gamma}+e^{\frac{2}{3}\Gamma}+e^{\frac{1}{3}\Gamma}-1}{e^{\Gamma}-1} \\
K_2 &= \frac{\gamma}{3}\frac{2-e^{-\frac{1}{3}\Gamma}}{1-e^{-\frac{1}{3}\Gamma}},
\end{aligned}
$$

*where* $\Gamma = \frac{\gamma}{\lambda}T$.

**Proof.** Equation (53) can be written in this context as equation (58), where $f$ is the density of $\Delta x$ and $C$ and $K$ are as defined. For values of $C$ sufficiently large or values of $K$ sufficiently small, equation (58) will be satisfied and there will exist a $\widetilde{\Delta x}$ such that cooperation is possible. For existence of $\overline{\Delta x}$, note that zero is a solution. The left-hand side of equation 58 is bounded since $f$ has finite variance and the right-hand-side is unbounded. Hence the supremum of $\widetilde{\Delta x}$ is bounded. As long as $\Delta x < \widetilde{\Delta x}$, the traders will cooperate since the value of cooperating exceeds that of a one-time deviation and subsequent grim-trigger play. If $\Delta x \geq \widetilde{\Delta x}$, the traders will predate and resume cooperation in the next period if possible. If equation 58 is not satisfied, then cooperation is not possible and the traders will always predate. Thus, there exists a subgame perfect Nash equilibrium as described. ■

Figure 4 shows an example of existence with a normally distributed shock. The left-hand-side of the inequality is plotted with the solid-line and the right-hand-side is plotted with the dashed-line. For $\widetilde{\Delta x}$ in an interval $[\underline{\Delta x}, \overline{\Delta x}]$, it is possible to sustain the subgame-perfect Nash equilibrium. For $\widetilde{\Delta x} > \overline{\Delta x}$, the value gained for deviation is too high, and cooperation cannot be maintained. The supremum $\overline{\Delta x}$ defines the most profitable strategy for the cartel. (Note

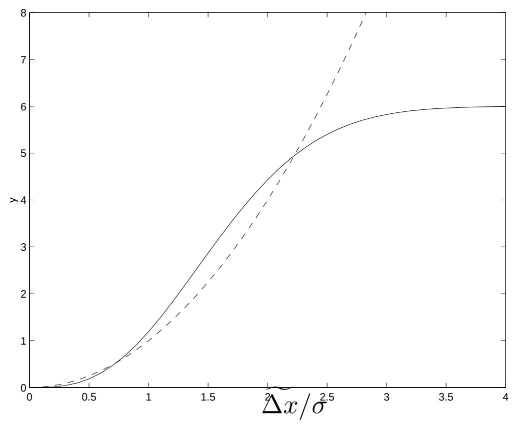


Figure 4: Left- (solid) and right-hand-side (dashed) of eq. (58). The curves intersect at zero, $\underline{\Delta x}$, and $\overline{\Delta x}$.

that $\overline{\Delta x}$ might be zero, in which case traders never cooperate.)

The nature of the solutions is essentially independent of the scale parameter of the distribution. Consider a family of distributions $f_a(y) = af(ay)$. If we consider solutions in terms of $\widetilde{\Delta x}/a$, the set of feasible thresholds is independent of the asset parameters (here $C$ and $K$). The inequality is equivalent to

$$2\frac{C}{K}\int_0^{\frac{\widetilde{\Delta x}}{a}} y^2 f_a(y)dy \geq \left(\frac{\widetilde{\Delta x}}{a}\right)^2, \tag{59}$$

so that, after the corresponding scaling, the solutions to the inequality are constant with scaling of the distribution.

As an example, consider the zero-mean normal distribution

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{y^2}{\sigma^2}}. \tag{60}$$

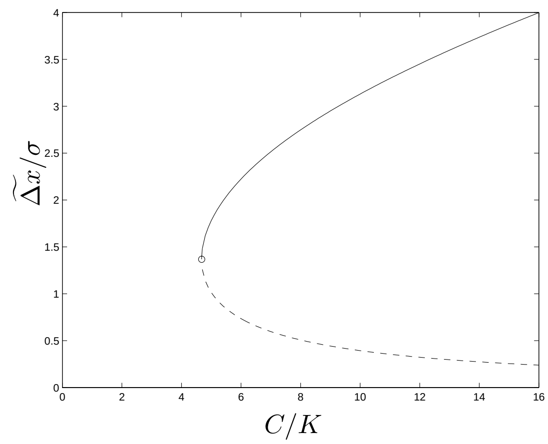


Figure 5: Normally distributed shocks, upper and lower bounds for $\widetilde{\Delta x}/\sigma$ as a function of the asset parameters.

Rewriting the inequality as

$$\frac{K}{C} \leq 2 \left( \frac{\widetilde{\Delta x}}{\sigma} \right)^{-2} \int_0^{\frac{\Delta x}{\sigma}} y^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy, \tag{61}$$

we can represent any such problem by parameterizing over $C/K$ and $\widetilde{\Delta x}/\sigma$. Figure 5 plots $\underline{\Delta x}/\sigma$ and $\overline{\Delta x}/\sigma$ as a function of $C/K$. The solid-line represents $\overline{\Delta x}/\sigma$ and the dotted-line represents $\underline{\Delta x}/\sigma$. For any value of $C/K$, the vertical segment between the lines is the set of $\widetilde{\Delta x}$ (in standard deviations) such that cooperation is possible. Note that there exists a critical value for $C/K$ (represented by the small circle), below which it is impossible to support cooperation because the gains from deviation are too great. Figure 6 shows the same analysis for the case in which the underlying distribution is double-sided exponential according to $f(y) = \frac{1}{a} e^{-\frac{y}{a}}$.

We now determine the minimum $C/K$ for each of the distributions for which

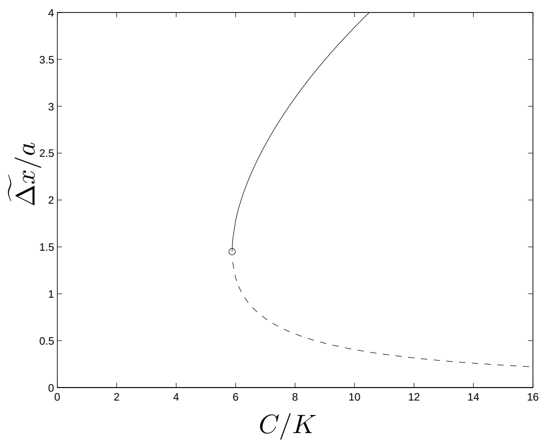


Figure 6: Double-exponentially distributed shocks, upper and lower bounds for $\widetilde{\Delta x}/a$ as a function of the asset parameters.

there is a non-zero $\overline{\Delta x}$, that is for which an episodic predation strategy is feasible. Consider first the normal distribution and the inequality in equation (61). Taking the derivative of the difference between the two sides of the inequality and equating to zero leads to

$$\frac{K}{C} = \frac{1}{\sqrt{2\pi}} \frac{\widetilde{\Delta x}}{\sigma} e^{-\frac{1}{2}\left(\frac{\widetilde{\Delta x}}{\sigma}\right)^2}. \tag{62}$$

Using this at the supremum (i.e., with equality holding), and after change of variable in the integral, we obtain

$$\int_0^{\frac{\widetilde{\Delta x}}{\sigma}} y^2 \frac{1}{2\pi} e^{-\frac{1}{2}y^2} dy = \frac{1}{2} \left(\frac{\widetilde{\Delta x}}{\sigma}\right)^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\widetilde{\Delta x}}{\sigma}\right)^2}, \tag{63}$$

which is straightforward to solve numerically for $\widetilde{\Delta x}/\sigma$. Since $C/K$ only depends on $\widetilde{\Delta x}$ through $\widetilde{\Delta x}/\sigma$, the minimum $C/K$ ratio for which there is a feasible strategy of the episodic predation type does not depend on the scale parameter of the distribution. For the normal distribution, the minimum $C/K$ for which there is a non-zero $\overline{\Delta x}$, that is for which an episodic predation strategy is feasible, is $C/K = 4.6729$ for any $\sigma$. The threshold for this $C/K$ ratio is $\underline{\Delta x} = \overline{\Delta x} = 1.3688\,\sigma$. The same analysis can be performed for other distributions. For the double-sided exponential distribution, $f(y) = \frac{1}{a}e^{-ay}$, the results as above are $\frac{C}{K} = 5.8824$. and $\underline{\Delta x} = \overline{\Delta x} = 1.4512\,a$.

## 3 B Contagion across markets

Suppose that the members of the oligopoly can cooperate in more than one market. For example, consider institutional traders who dominate mortgage markets are also strategic traders in other fixed income markets. If a liquidity event is large enough to disturb cooperation in one market, it may also affect cooperation in the others. According to Bernheim and Whinston (1990), if markets are not identical, multimarket contact supports cooperation. In our case, and since most assets are not perfectly correlated, multi-market contact makes it easier to maintain cooperation. In this section, we first evaluate the contagion of illiquidity using the repeated game in Section 2 with $\Delta x$ fixed. Then, we demonstrate the effects of multi-market contact on the episodic illiquidity that occurs across markets.

To extend the model of Section 2 to multiple markets, define the probabilities $p_{00}^j$, $p_{10}^j$, $p_{01}^j$, and $p_{11}^j$ as before, each associated with the $j^{th}$ market. We also define $\Delta x_j$ as the amount of asset $j$ required to be traded and $V_p^j$ and $V_d^j$ as the values to be gained by the predator and distressed seller in the $j^{th}$ market.

**Result 12** *(Multimarket Contact) The $\delta$ required to support cooperation in the* *presence of multimarket contact is*

$$\delta \geq \frac{\sum_{j=1}^{n} V_p^j}{\sum_{j=1}^{n} p_{10}^j [V_1(\Delta x_j) - V_d^j] + \frac{1}{2} p_{11}^j [V_1(2\Delta x_j) - V_2(2\Delta x_j)] + (1 - p_{01}^j) V_p^j}. \tag{64}$$

**Proof.** Proceed as for Result 9. ■

The key and easily observed point is that the restrictions required to maintain stability will now be less stringent. When there is more at stake and the penalties for deviation are greater, it is easier to support the Pareto superior equilibrium. However, once a significant liquidity event occurs, or if the parameters change substantially, multiple markets may quickly become volatile.

We now consider episodic illiquidity across $n$ markets, where the trading requirement in each market is a stochastic random variable. We define a liquidity event to be such that all trading targets for each of the $n$ assets have the same sign (i.e. liquidity shocks occur in the same direction in all markets). The trading targets are modeled as jointly normal and conditionally independent given that they are either all positive or all negative. We also use the simplifying assumption that trading targets in all the assets have the same variance. The density in the positive orthant ($y$ such that $y_i \geq 0$, all $i$) and in the negative orthants ($y$ such that $y_i \leq 0$, all $i$) is

$$f(y) = \frac{2^{n-1}}{\sigma^n (2\pi)^{n/2}} e^{-\frac{y^T y}{\sigma^2}}, \tag{65}$$

and zero elsewhere.

The shape of the optimal region for cooperation is spherical. This is the region in which the temptation to predate, which is proportional to $\sum_{i=1}^{n} \Delta x_i^2$, is constant.

The inequality for $n$ assets involves an integral in $n$ dimension which, using the radial symmetry of the normal distribution can be written as

$$2C \int_0^r S_n y^{n+1} f(y) dy \geq K r^2, \tag{66}$$

where $r$ is the radius of the cooperation region, and

$$S_n = \frac{1}{2^n} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \tag{67}$$

is the area of the intersection of the sphere of unit radius in $n$ dimension with the positive orthant. It can easily be verified that, as for the one-asset case, the nature of the solutions is essentially independent of the scale parameter of the distribution (the standard deviation).

Figure 7 is the multi-market version of Figure 5 in that it plots $\underline{\Delta x}$ and $\overline{\Delta x}$ for episodic illiquidity over $n$ markets, $n = 1, 2, \ldots, 8$. The minimum value of $C/K$ that is required to support cooperation decreases as the number of markets increases. Adding markets can make cooperation possible when it does not exist. (Consider, for instance, $C/K = 4.0$. With these parameters, traders are unable to cooperate over one market, but are able to do so over 2 or more markets.) Also note that $r$ increases with $n$. The probability that an episode of predation will occur is in fact seen to decrease with $n$. Episodes of predation will be more significant, since they now affect $n$ markets, but less frequent with contagion strategies.

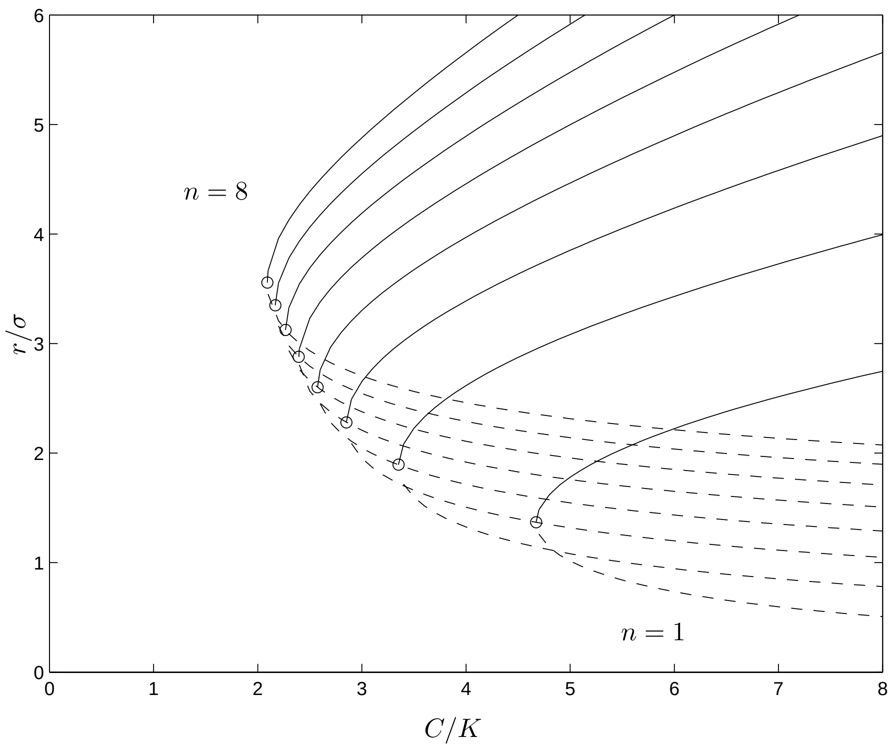


Figure 7: Lowest and highest $r$ such that the strategy (episodic predation with contagion over $n$ assets, $n = 1, 2, \ldots, 8$) is an equilibrium, plotted as a function of the asset parameters, common to all assets. The shocks are independent and normally distributed conditional on shocks being either all positive or all negative.

| $n$ | $C/K$ | $n$ | $C/K$ | $n$ | $C/K$ | $n$ | $C/K$ |
|---|---|---|---|---|---|---|---|
| 1 | 4.6729 | 6 | 2.2664 | 11 | 1.9299 | 16 | 1.7740 |
| 2 | 3.3509 | 7 | 2.1687 | 12 | 1.8907 | 17 | 1.7517 |
| 3 | 2.8507 | 8 | 2.0913 | 13 | 1.8563 | 18 | 1.7314 |
| 4 | 2.5747 | 9 | 2.0280 | 14 | 1.8259 | 19 | 1.7127 |
| 5 | 2.3950 | 10 | 1.9751 | 15 | 1.7986 | 20 | 1.6954 |

The minimum values of $C/K$ for cooperation over multiple markets, $n = 1, 2, \ldots, 20$, are listed in the table above.

## 4 Conclusion

We solve a competitive trading game by posing a continuous-time, dynamic programming problem for our traders, using an asset pricing equation which accounts for transaction costs. According to our model, traders 'race' to market, selling quickly in the beginning of the period. In the equilibrium strategy traders sell-off at a decreasing exponential rate. Also in equilibrium, predators initially race distressed traders to market, but eventually 'fade' them and buy back. The presence of predators in the market leads to a surplus loss to liquidity providers in the market.

We model cooperation by embedding this predatory stage game in an infinitely-repeated game. Cooperation allows for the trading of large blocks of the asset at more favorable prices. We show how traders can cooperate to avoid the surplus loss due to predatory trading and provide predictions as to what magnitude of liquidity event is required to trigger an observable shock in the market.

The breakdown of cooperation can lead to episodic illiquidity in the market. We describe an equilibrium strategy where traders cooperate most of the time through repeated interaction, providing 'apparent liquidity' to each other. However, episodically this cooperation breaks down, especially when the stakes are high, leading to opportunism and loss of this apparent liquidity. We believe that our model explains why episodic liquidity breakdowns do not occur more often.

In conclusion, we believe our work presents a strong argument for the level of predation or cooperation in financial markets being a determinant of the amount of liquidity available, and a factor causing the episodic nature of illiquidity.

# References

Abreu, D. (1988). On the theory of infinitely repeated games with discounting. *Econometrica 56*(2), 383–396.

Attari, M., A. Mello, and M. Ruckes (2004). Arbitraging arbitrageurs. Forthcoming Journal of Finance.

Battalio, R., A. Ellul, and R. Jennings (2004). Reputation effects in trading on the new york stock exchange.

Berhardt, D., V. Dvoracek, E. Hughson, and I. Werner (2004). Why do larger orders receive discounts on the london stock exchange? Ohio State University, unpublished.

Bernheim, B. and M. Whinston (1990). Multimarket contact and collusive behavior. *RAND Journal of Economics 21*(1), 1–26.

Bertsimas, D. and A. Lo (1998). Optimal control of execution costs. *Journal of Financial Markets 1*, 1–50.

Brunnermeier, M. and L. Pedersen (2004). Predatory trading. Forthcoming Journal of Finance.

Chan, L. and J. Lakonishok (1995). The behavior of stock prices around institutional trades. *Journal of Finance 50*(4), 1147–1174.

Cocco, J., F. Gomes, and N. Martins (2003). Lending relationships in the interbank market. London School of Business, unpublished.

DeMarzo, P. and B. Uroševic (2000). Optimal trading by a large shareholder. Stanford Graduate School of Business and Haas School of Business, unpublished.

Desgranges, G. and T. Foucault (2002). Reputation-based pricing and price improvements in dealership markets. Universite de Cergy, unpublished.

Dutta, P. and A. Madhavan (1997). Competition and collusion in dealer markets.

*Journal of Finance 52*(1), 245–276.

Fedyk, Y. (2001). Large investor optimization problem with direct eect of trades on prices. Olin School of Business, unpublished.

Foster, F. and S. Viswanathan (1996). Strategic trading when agents forecast the forecasts of others. *Journal of Finance 51*(4), 1437–1478.

Gabaix, X., A. Krishnamurthy, and O. Vigneron (2004). Limits of arbitrage: Theory and evidence from the mortgage-backed securities market. MIT, unpublished.

Gennotte, G. and A. Kyle (1991). Intertemporal insider trading with a smooth order flow. Haas School of Business Administration, Berkeley, unpublished.

Hansch, O., N. Naik, and S. Viswanathan (1999). Preferencing, internalization, best execution, and dealer profits. *Journal of Finance 54*(5), 1799–1828.

Holthausen, R., R. Leftwich, and D. Mayers (1990). Large block transactions, the speed of response, and temporary and permanent stock-price effects. *Journal of Financial Economics 26*(1), 71–95.

Kaul, A., V. Mehrotra, and R. Morck (2000). Demand curves for stocks do slope down: New evidence from an index weights adjustment. *Journal of Finance 55*(2), 893–912.

Keim, D. and A. Madhavan (1996). The upstairs market for large-block transactions: Analysis and measurement of price effects. *Review of Financial Studies 9*(1), 1–36.

Massa, M. and A. Simonov (2003). Reputation and interdealer trading: A microstructure analysis of the treasury bond market. *Journal of Financial Markets 6*(6), 99–141.

Pastor, L. and R. Stambaugh (2003). Liquidity risk and expected stock returns. *Journal of Political Economy 3*, 642–685.

Pritsker, M. (2004). Large investors: Implications for equilibrium asset returns, shock absorption, and liquidity. Federal Reserve, unpublished.

Ramadorai, T. (2003). Persistence, performance, and prices in foreign exchange markets. Harvard University, unpublished.

Reiss, P. and I. Werner (2003). Anonymity, adverse selection and the sorting of interdealer trades. Ohio State University, unpublished.

Rotemberg, J. and G. Saloner (1986). A supergame-theoretic model of price wars during booms. *American Economic Review 76*(3), 390–407.

Vayanos, D. (1998). Transaction costs and asset prices: A dynamic equilibrium model. *Review of Financial Studies 11*(1), 1–58.