Functions of Two Variables

Definition of a function of two variables

Until now, we have only considered functions of a single variable, $y = f(x)$. However, many real-world functions consist of two (or more) variables. E.g., the area function of a rectangular shape depends on both its width and its height. And, the pressure of a given quantity of gas varies with respect to the temperature of the gas and its volume. We define a function of two variables as follows:

A function *f* of two variables is a relation that assigns to every ordered pair of input values (x, y) in a set called the *domain* a unique output value denoted by $f(x, y)$. The set of output values is called the *range*.

Since the domain consists of ordered pairs, we may consider the domain to be all (or part) of the *x*-*y* plane.

Unless otherwise stated, we will assume that the variables *x* and *y* and the output value $f(x, y)$ represent real numbers.

Determine if the following relations are functions.

1. The area *z* of a rectangle is the product of its length *x* and its width *y*, $z = f(x, y) = xy$.
2. $z^2 = x^2 + y^2$.

Domain of a function of two variables

As we did for functions of a single variable, we will assume that the domain of the function consists of all real numbers unless a restriction is stated or this is impossible.

Determine the domain of the function, $z = f(x, y) = \sqrt{x - y}$. GRAPH domain.

Range of a function of two variables

Finding the range of a function of two variables is not always easy, but sometimes we can tell without too much difficulty.

Determine the range of the function, $z = f(x, y) = x^2 + y^2 + 1$.

Calculating values of functions of two variables

Let's calculate some output values for a function of two variables.

Let $z = f(x, y) = x^2 y$. Calculate

1. $f(-2, 3)$
2. $\dfrac{f(x+h, y) - f(x, y)}{h}$

Definition of graph of function of two variables

Suppose that *f* is a function of *x* and *y*. The *graph* of *f* consists of all points in *space* of the form $(x, y, f(x, y))$ where (x, y) is in the domain of *f*.

The graph of a function of two variables is a surface in three dimensions.

Constructing the graph of a function of two variables

Generally, it is difficult to draw or visualize a surface in three dimensions. To help us draw surfaces, we will draw *contour lines* and *cross sections* of the surface.

A *contour line* of a surface is a graph in three dimensions where the value $z = f(x, y)$ is held constant.

We may consider a contour line as the intersection of a plane and a surface in three dimensions, where the plane is parallel to the *x*-*y* plane.

Consider a situation where a family's weekly income consists of the sum of the husband's and wife's salaries. If the husband earns \$5 per hour and works *x* hour per week, and the wife earns \$10 per hour and works *y* hours per week, then the family's weekly income is $z = f(x,y) = 5x + 10y$, where the domain is $x \geq 0$ and $y \geq 0$. Our goal is to graph this function in three dimensions. We begin by drawing some contour lines.

Let $z = 200$, then the family's weekly income is \$200, and $5x + 10y = 200$. This is a linear equation. So we can conclude that all (x,y) that satisfy $5x + 10y = 200$ are contained on the line segment connecting the points $(0,20)$ and $(40,0)$. Remember that for these points $z = 200$ on the contour line. This graph is a line segment in three dimensions. Let's plot some more contour lines.

Let $z = 300$, then the family's weekly income is \$300, and $5x + 10y = 300$. All (x,y) that satisfy $5x + 10y = 300$ are contained on the line segment connecting the points $(0,30)$ and $(60,0)$. Label the contour line $z = 300$.

Let $z = 400$, then the family's weekly income is \$400, and $5x + 10y = 400$. All (x,y) that satisfy $5x + 10y = 400$ are contained on the line segment connecting the points $(0,40)$ and $(80,0)$. Label the contour line $z = 400$.

By placing all of these graphs together on the same *x*-*y* axes and clearly labeling the value of *z* with each graph, we begin to understand the graph (surface) of our function in three dimensions.

Another aid to drawing graphs of functions of two variables is to sketch the *cross sections* of the surface.

A *cross section* of a surface is a graph in three dimensions where the value of either the *x* or *y* variables is held constant.

Consider our earlier problem where $z = f(x,y) = 5x + 10y$.

Draw the cross section where $x = 0$. Note the graph of $x = 0$ is, in fact, the *yz* plane. The cross section of $z = f(x,y) = 5x + 10y$ where $x = 0$ is the line $z = f(x,y) = 10y$ in the *yz* plane.

Draw the cross section where $y = 0$. Note the graph of $y = 0$ is the *xz* plane. The cross section of $z = f(x,y) = 5x + 10y$ where $y = 0$ is the line $z = f(x,y) = 5x$ in the *xz* plane.

Since the domain of *z* consisted of all $x \geq 0$ and $y \geq 0$, we may consider these two cross sections as limiting lines of our surface. The graph (surface) of *z* consists of all points on contour lines between these two lines in the *first* octant of the *xyz* space.

<u>Rate of change in the *x* and *y* directions</u>

We can also consider the rate of change of a function of two variables. However, since we have two variables, we can consider *two rates of change*: a rate of change in the *x* direction and a rate of change in the *y* direction.

Consider our earlier problem where $z = f(x,y) = 5x + 10y$.

What is the rate of change of *z* in the *x* direction? For each increase of one unit in *x*, the value of *z* increases by 5. Therefore, the rate of change in the *x* direction is $\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \frac{5}{1} = 5$. And, the rate of change in the *y* direction is $\lim_{\Delta y \to 0} \frac{\Delta z}{\Delta y} = \frac{10}{1} = 10$.