Chaos & Fractals in Financial Markets

by J. Orlin Grabbe

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Part 1

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Prologue: The Rolling of the Golden Apple

In 1776, a year in which political rebels in Philadelphia were proclaiming their independence and freedom, a physicist in Europe was proclaiming total dependence and determinism. According to Pierre-Simon **Laplace**, if you knew the initial conditions of any situation, you could determine the future far in advance: "The present state of the system of nature is evidently a consequence of what it was in the preceding moment, and if we conceive of an intelligence which at a given instant comprehends all the relations of the entities of this universe, it could state the respective positions, motions, and general effects of all these entities at any time in the past or future."

The Laplacian universe is just a giant pool table. If you know where the balls were, and you hit and bank them correctly, the right ball will always go into the intended pocket.

Laplace's hubris in his ability (or that of his "intelligence") to forecast the future was completely consistent with the equations and point of view of classical mechanics. Laplace had not encountered nonequilibrium thermodynamics, quantum physics, or chaos. Today some people are frightened by the very notion of chaos. (I have explored this at length in an essay devoted to chaos from a philosophical perspective. But the same is also true with respect to the somewhat related mathematical notion of chaos.) Today there is no justification for a Laplacian point of view.

At the beginning of this century, the mathematician Henri Poincaré, who was studying planetary motion, began to get an inkling of the basic problem:

> "It may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible" (1903).

In other words, he began to realize "deterministic" isn't what it's often cracked up to be, even leaving aside the possibility of other, nondeterministic systems. An engineer might say to himself: "I know where a system is now. I know the location of this (planet, spaceship, automobile, fulcrum, molecule) almost precisely. Therefore I can predict its position X days in the future with a margin of error precisely related to the error in my initial observations."

Yeah. Well, that's not saying much. The prediction error may explode off to infinity at an exponential rate (read the discussion of *Lyapunov exponents* later). Even God couldn't deal with the margin of error, if the system is chaotic. (There is no omniscience. Sorry.) And it gets even worse, if the system is nondeterministic.

The distant future? You'll know it when you see it, and that's the first time you'll have a clue. (This statement will be slightly modified when we discuss a system's *global* properties.)

I Meet Chaos

I first came across something called "dynamical systems" while I was at the University of California at Berkeley. But I hadn't paid much attention to them. I went through Berkeley very fast, and didn't have time to screw around. But when I got to Harvard for grad school, I bought René Thom's book *Structural Stability and Morphogenesis*, which had just come out in English. The best part of the book was the photos.

Consider a **crown** worn by a king or a princess, in fairy tales or sometimes in real life. Why does a crown look the way it does? Well, a crown is kind of round, so it will fit on the head, and it has spires on the rim, like little triangular hats—but who knows why—and sometimes on the end of the spires are little round balls, jewels or globs of gold. Other than the requirement that it fit on the head, the form of a crown seems kind of arbitrary.

But right there in Thom's book was a photo of a steel ball that had been dropped into molten lead, along with the reactive splash of the molten liquid. The lead splash was a *perfect crown*--a round vertical column rising upward, then branching into triangular spires that get thinner and thinner (and spread out away from the center of the crown) as you approached the tips, but instead of ending in a point, each spire was capped with a spherical blob of lead. In other words, the shape of a crown isn't arbitrary at all: under certain conditions its form occurs spontaneously whenever a sphere is dropped into liquid. So the king's crown wasn't created to "symbolize" this or that. The form came first, a natural occurrence, and the interpretation came later.

The word "morphogenesis" refers to the forms things take when they grow: bugs grow into a particular shape, as do human organs. I had read a number of books on general systems theory by Ervin Laszlo and Ludwig von Bertalanffy, which discuss the concepts of morphogenesis, so I was familiar with the basic ideas. Frequent references were made to biologist D'Arcy Thompson's book *On Growth and Form*. But it was only much later, when I began doing computer art, and chaotically created a more or less perfectly formed *ant* by iterating a fifth-degree complex equation (that is, an equation containing a variable z raised to the fifth power, z^5, where z is a complex number, such as $z = .5 + 1.2$ sqrt(-1)), that I really understood the power of the idea. If the shape of ants is arbitrary, then why in the hell do they look like fifth-degree complex equations?

Anyway, moving along, in grad school I was looking at the forms taken by asset prices, foreign exchange rates in particular. A foreign exchange rate is the price that one fiat currency trades for another. But I could have been looking at stock prices, interest rates, or commodity prices—the principles are the same. Here the assumption is that the systems generating the prices are nondeterministic (stochastic, random)—but that doesn't prevent there being hidden form, hidden order, in the shape of *probability distributions*.

Reading up on price distributions, I came across some references to Benoit Mandelbrot. Mandelbrot, an applied mathematician, had made a splash in economics in the early-1960s with some heretical notions of the probabilities involved in price distributions, and had acquired as a disciple Eugene Fama [1] at the University of Chicago. But then Fama abandoned this heresy (for alleged empirical reasons that I find manifestly absurd), and everyone breathed a sigh of relief and returned to the familiar world of least squares, and price distributions that were *normal* (as they believed) in the *social* sense as well as the *probability* sense of a "normal" or Gaussian distribution.

In economics, when you deal with prices, you first take logs, and then look at the changes between the logs of prices [2]. The changes between these log prices are what are often referred to as the *price distribution*. They may, for example, form a Bell-shaped curve around a mean of zero. In that case, the changes between logs would have a normal (Gaussian) distribution, with a mean of zero, and a standard deviation of whatever. (The actual prices themselves would have a *lognormal* distribution. But that's not what is meant by "non-normal" in most economic contexts, because the usual reference is to *changes in the logs* of prices, and not to the actual prices themselves.)

At the time I first looked at *non*-normal distributions, they were very much out of vogue in economics. There was even active hostility to the idea there could be such things in real markets. Many people had their nice set of tools and results that would be threatened (or at least they thought would be threatened) if you changed their probability assumptions. Most people had heard of Mandelbrot, but curiously no one seemed to have the slightest clue as to what the actual details of the issue were. It was like option pricing theory in many ways: it wasn't taught in economic departments at the time, because none of the professors understood it.

I went over to the Harvard Business School library to read Mandelbrot's early articles. The business school library was better organized than the library at the Economics Department, and it had a better collection of books and journals, and it was extremely close to where I lived on the Charles River in Cambridge. In one of the articles, Mandelbrot said that the ideas therein were first presented to an economic audience in Hendrik Houthakker's international economics seminar at Harvard. Bingo. I had taken international finance from Houthakker and went to talk to him about Mandelbrot. Houthakker had been a member of Richard Nixon's Council of Economic Advisors, and was famous for the remark: "[Nixon] had no strong interest in international economic affairs, as shown by an incident recorded on the Watergate tapes where Haldeman comes in and wants to talk about the Italian lira. His response was '[expletive deleted] the Italian lira!'"

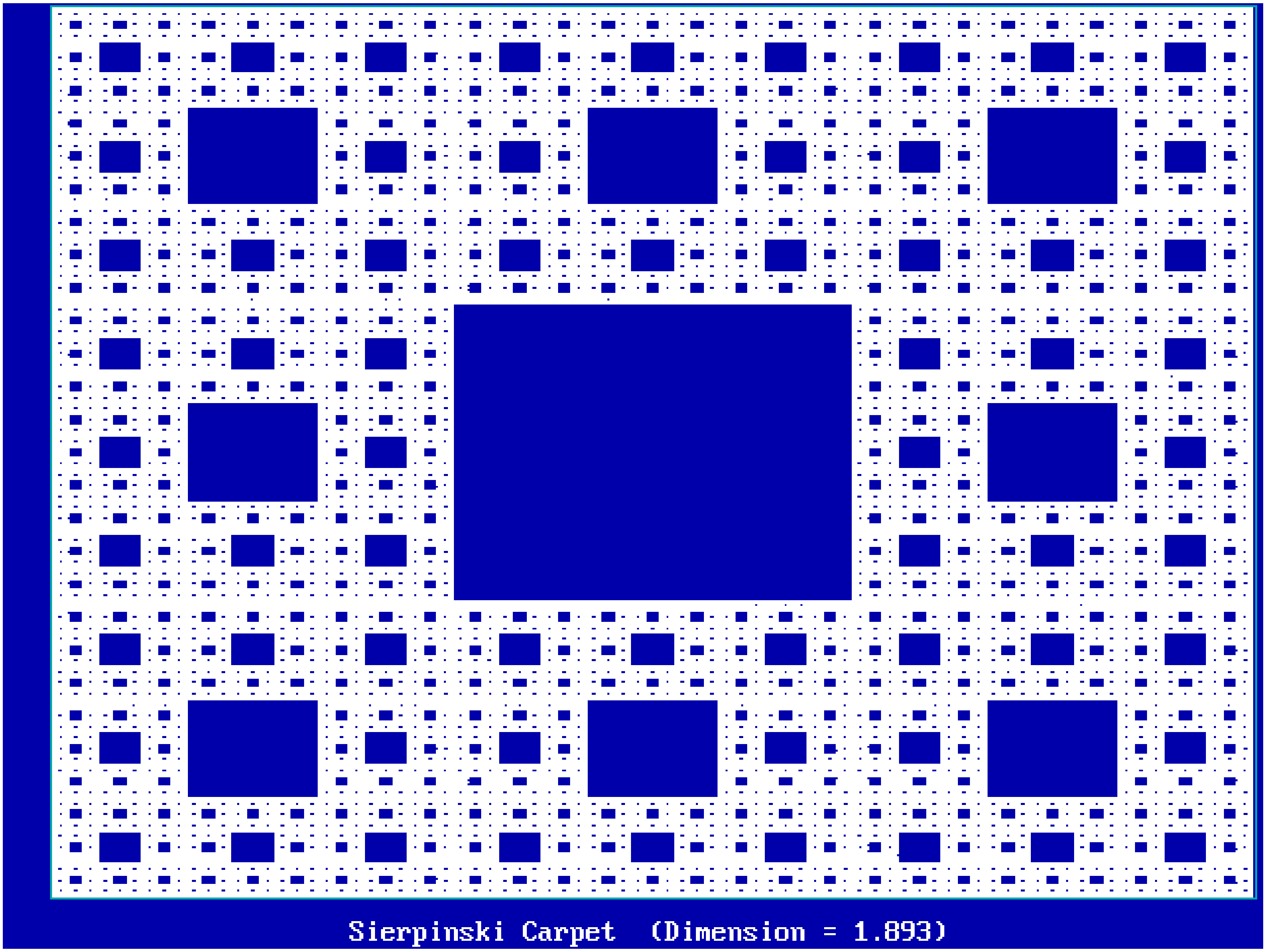
Houthakker told me he had studied the distribution of cotton futures prices and didn't believe they had a normal distribution. He had given the same data to Mandelbrot. He told me Mandelbrot was back in the U.S. from a sojourn in France, and that he had seen him a few weeks previously, and Mandelbrot had a new book he was showing around. I went over to the Harvard Coop (that's pronounced "coupe" as in "a two-door coupe", no French accent) and found a copy of Mandelbrot's book.

Great photos! That's when I learned what a fractal was, and ended up writing two of the three essays in my PhD thesis on fractal price distributions [3].

Fractals led me back into chaos, because maps (graphics) of chaos equations create fractal patterns.

Preliminary Pictures and Poems

The easiest way to begin to explain an elephant is to first show someone a picture. You point and say, "Look. Elephant." So here's a picture of a **fractal**, something called a Sierpenski carpet [4]:



Sierpinski Carpet (Dimension = 1.893)

Notice that it has a solid blue square in the center, with 8 additional smaller squares around the center one.

1	2	3

8	center square	4
7	6	5

Each of the 8 smaller squares looks just like the original square. Multiply each side of a smaller square by 3 (increasing the area by 3 x 3 = 9), and you get the original square. Or, doing the reverse, divide each side of the original large square by 3, and you end up with one of the 8 smaller squares. At a scale factor of 3, all the squares look the same (leaving aside the disgarded center square).

You get 8 copies of the original square at a scale factor of 3. Later we will see that this defines a fractal dimension of log 8 / log 3 = 1.8927. (I said *later*. Don't worry about it now. Just notice that the dimension is not a nice round number like 2 or 3.)

Each of the smaller squares can also be divided up the same way: a center blue square surrounded by 8 *even smaller* squares. So the original 8 small squares can be divided into a total of 64 *even smaller* squares—each of which will look like the original big square if you multiply its sides by 9. So the fractal dimension is log 64 / log 9 = 1.8927. (You didn't expect the dimension to change, did you?) In a factal, this process goes on forever.

Meanwhile, without realizing it, we have just defined a *fractal* (or *Hausdorff*) *dimension*. If the number of small squares is **N** at a scale factor of **r**, then these two numbers are related by the fractal dimension **D**:

$\mathbf{N = r^D}$.

Or, taking logs, we have D = log N / log r.

The same things keep appearing when we scale by r, because the object we are dealing with has a fractal dimension of D.

Here is a poem about **fractal fleas**:

> **Great fleas have little fleas, upon their backs to bite 'em**
> **And little fleas have lesser fleas, and so *ad infinitum*,**
> **And the great fleas themselves, in turn, have greater fleas to go on,**
> **While these again have greater still, and greater still, and so on.**

Okay. So much for a preliminary look at fractals. Let's take a preliminary look at chaos, by asking what a dynamical system is.

Dynamical Systems

What is a *dynamical system*? Here's one: **Johnny grows 2 inches a year.** This system explains how Johnny's height changes over time. Let x(n) be Johnny's height this year. Let his height next year be written as x(n+1). Then we can write the dynamical system in the form of an equation as:

$x(n+1) = x(n) + 2$.

See? Isn't math simple? If we plug Johnny's current height of x(n) = 38 inches in the right side of the equation, we get Johnny's height next year, x(n+1) = 40 inches:

$x(n+1) = x(n) + 2 = 38 + 2 = 40$.

Going from the right side of the equation to the left is called an *iteration*. We can iterate the equation again by plugging Johnny's new height of 40 inches into the right side of the equation (that is, let x(n)=40), and we get x(n+1) = 42. If we iterate the equation 3 times, we get Johnny's height in 3 years, namely 44 inches, starting from a height of 38 inches).

This is a *deterministic* dynamical system. If we wanted to make it *nondeterministic* (*stochastic*), we could let the model be: **Johnny grows 2 inches a year, more or less**, and write the equation as:

$x(n+1) = x(n) + 2 + e$

where *e* is a small error term (small relative to 2), and represents a drawing from some probability distribution.

Let's return to the original deterministic equation. The original equation, x(n+1) = x(n) + 2, is ***linear***. Linear means you either add variables or constants or multiply variables by constants. The equation

$z(n+1) = z(n) + 5\, y(n) - 2\, x(n)$

is linear, for example. But if you *multiply variables* together, or *raise them to a power* other than one, the equation (system) is *nonlinear*. For example, the equation

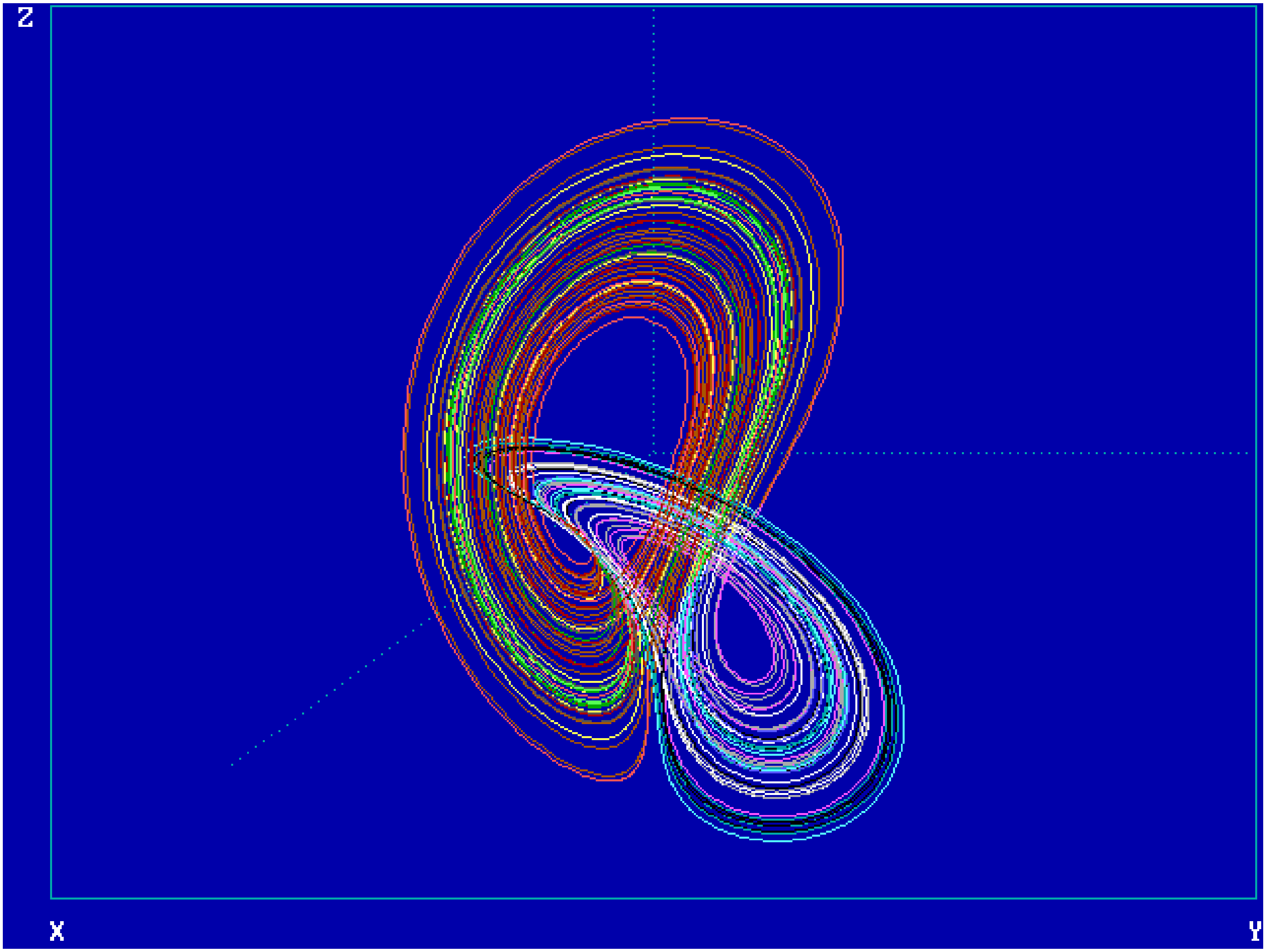
$x(n+1) = x(n)^2$

is nonlinear because x(n) is squared. The equation

$z = xy$

is nonlinear because two variables, **x** and **y**, are multiplied together.

Okay. Enough of this. What is chaos? Here is a picture of chaos. The lines show how a dynamical system (in particular, a *Lorenz system*) changes over time in three-dimensional space. Notice how the line (path, trajectory) loops around and around, never intersecting itself.



Notice also that the system keeps looping around two general areas, as though it were drawn to them. The points from where a system feels compelled to go in a certain direction are called the *basin of attraction*. The place it goes to is called the *attractor*.

Here's an equation whose attractor is a single point, zero:

x(n+1) = .9 x(n) .

No matter what value you start with for x(n), the next value, x(n+1), is only 90 percent of that. If you keep iterating the equation, the value of x(n+1) approaches zero. Since the attractor in this case is only a single point, it is called a *one-point attractor*.

Some attractors are simple circles or odd-shaped closed loops—like a piece of string with the ends connected. These are called *limit cycles*.

Other attractors, like the Lorenz attractor above, are really weird. Strange. They are called *strange attractors.*

Okay. Now let's define chaos.

What is Chaos?

What are the characteristics of chaos? First, chaotic systems are *nonlinear* and follow trajectories (paths, highways) that end up on non-intersecting loops called *strange attractors*. Let's begin by understanding what these two terms mean.

I am going to repeat some things I said in the previous section. Déjà vu. But, as in the movie *The Matrix*, déjà vu can communicate useful information. All over again.

Classical systems of equations from physics were *linear*. Linear simply means that outputs are *proportional* to inputs. Proportional means you either multiply the inputs by constants to get the output, or add a constant to the inputs to get the output, or both. For example, here is a simple linear equation from the capital-asset pricing model used in corporate finance:

$$\mathbf{E(R)} = \alpha + \beta\ \mathbf{E(Rm)}.$$

It says the expected return on a stock, **E(R)**, is proportional to the return on the market, **E(Rm)**. The input is **E(Rm)**. You multiply it by β ("beta"), then add α ("alpha") to the result—to get the output **E(R)**. This defines a linear equation.

Equations which *cannot* be obtained by multiplying isolated variables (not raised to any power except the first) by constants, and adding them together, are nonlinear. The equation $y = x^2$ is nonlinear because it uses a power of two: namely, x squared. The equation z = 4xy-10 is nonlinear because a variable x is multipled by a variable y.

The equation z = 5+ 3x-4y-10z is linear, because each variable is multiplied only by a constant, and the terms are added together. If we multiply this last equation by 7, it is still linear: 7z = 35 + 21x – 28y – 70z. If we multiply it by the variable **y**, however, it becomes nonlinear: $zy = 5y + 3xy-4y^2-10zy$.

The science of chaos looks for characteristic patterns that appear in complex systems. Unless these patterns were exceedingly simple, like a single equilibrium point ("the equilibrium price of gold is $300"), or a simple closed or oscillatory curve (a circle or a sine wave, for example), the patterns are referred to as *strange attractors*.

Such patterns are traced out by self-organizing systems. Names other than strange attractor may be used in different areas of science. In biology (or sociobiology) one refers to *collective patterns* of animal (or social) behavior. In Jungian psychology, such patterns may be called *archetypes* [5].

The main feature of chaos is that simple deterministic systems can generate what appears to be random behavior. Think of what this means. On the good side, if we observe what appears to be complicated, random behavior, perhaps it is being generated by a few deterministic rules. And maybe we can discover what these are. Maybe life isn't so complicated after all. On the bad side, suppose we have a simple deterministic system. We may think we understand it— it looks so simple. But it may turn out to have exceedingly complex properties. In any case, chaos tells us that whether a given random-appearing behavior is at basis random or deterministic may be undecidable. Most of us already know this. We may have used random number generators (really *pseudo*-random number generators) on the computer. The "random" numbers in this case were produced by simple deterministic equations.

I'm Sensitive—Don't Perturb Me

Chaotic systems are *very sensitive to initial conditions*. Suppose we have the following simple system (called a *logistic equation*) with a single variable, appearing as input, x(n), and output, x(n+1):

$$\mathbf{x(n+1) = 4\ x(n)\ [1-x(n)].}$$

The input is x(n). The output is x(n+1). The system is nonlinear, because if you multiply out the right hand side of the equation, there is an $x(n)^2$ term. So the output is not proportional to the input. Let's play with this system. Let x(n) = .75. The output is

4 (.75) [1- .75] = .75.

That is, x(n+1) = .75. If this were an equation describing the price behavior of a market, the market would be in equilibrium, because today's price (.75) would generate the same price tomorrow. If x(n) and x(n+1) were expectations, they would be self-fulfilling. Given today's price of x(n) = .75, tomorrow's price will be x(n+1) = .75. The value .75 is called a ***fixed point*** of the equation, because using it as an input returns it as an output. It stays fixed, and doesn't get transformed into a new number.

But, suppose the market starts out at x(0) = .7499. The output is

4 (.7499) [1-.7499] = .7502 = x(1).

Now using the previous day's output x(1) = .7502 as the next input, we get as the new output:

4 (.7502) [1-.7502] = .7496 = x(2).

And so on. Going from one set of inputs to an output is called an ***iteration***. Then, in the next iteration, the new output value is used as the input value, to get another output value. The first 100 iterations of the logistic equation, starting with x(0) = .7499, are shown in Table 1.

Finally, we repeat the entire process, using as our first input x(0) = .74999. These results are also shown in **Table 1**. Each set of solution paths—x(n), x(n+1), x(n+2), etc.—are called *trajectories*. Table 1 shows three different trajectories for three different starting values of x(0).

Look at iteration number 20. If you started with x(0) = .75, you have x(20) = .75. But if you started with x(0) = .7499, you get x(20) = .359844. Finally, if you started with x(0) = .74999, you get x(20) = .995773. Clearly a small change in the intitial starting value causes a large change in the outcome after a few steps. The equation is very sensitive to initial conditions.

A meteorologist name Lorenz discovered this phenomena in 1963 at MIT [6]. He was rounding off his weather prediction equations at certain intervals from six to three decimals, because his printed output only had three decimals. Suddenly he realized that the entire sequence of later numbers he was getting were different. Starting from two nearby points, the trajectories diverged from each other rapidly. This implied that long-term weather prediction was impossible. He was dealing with chaotic equations.

Table 1: First One Hundred Iterations of the Equation
x(n+1) = 4 x(n) [1- x(n)] with Different Values of x(0).

x(0):	**.75000**	**.74990**	**.74999**
Iteration			
1	.7500000	.750200	.750020
2	.7500000	.749600	.749960
3	.7500000	.750800	.750080
4	.7500000	.748398	.749840
5	.7500000	.753193	.750320
6	.7500000	.743573	.749360

7	.7500000	.762688	.751279
8	.7500000	.723980	.747436
9	.7500000	.799332	.755102
10	.7500000	.641601	.739691
11	.7500000	.919796	.770193
12	.7500000	.295084	.707984
13	.7500000	.832038	.826971
14	.7500000	.559002	.572360
15	.7500000	.986075	.979056
16	.7500000	.054924	.082020
17	.7500000	.207628	.301170
18	.7500000	.658075	.841867
19	.7500000	.900049	.532507
20	.7500000	.359844	.995773
21	.7500000	.921426	.016836
22	.7500000	.289602	.066210
23	.7500000	.822930	.247305
24	.7500000	.582864	.744581
25	.7500000	.972534	.760720
26	.7500000	.106845	.728099
27	.7500000	.381716	.791883
28	.7500000	.944036	.659218
29	.7500000	.211328	.898598
30	.7500000	.666675	.364478
31	.7500000	.888878	.926535

32	.7500000	.395096	.272271
33	.7500000	.955981	.792558
34	.7500000	.168326	.657640
35	.7500000	.559969	.900599
36	.7500000	.985615	.358082
37	.7500000	.056712	.919437
38	.7500000	.213985	.296289
39	.7500000	.672781	.834008
40	.7500000	.880587	.553754
41	.7500000	.420613	.988442
42	.7500000	.974791	.045698
43	.7500000	.098295	.174440
44	.7500000	.354534	.576042
45	.7500000	.915358	.976870
46	.7500000	.309910	.090379
47	.7500000	.855464	.328843
48	.7500000	.494582	.882822
49	.7500000	.999883	.413790
50	.7500000	.000470	.970272
51	.7500000	.001877	.115378
52	.7500000	.007495	.408264
53	.7500000	.029756	.966338
54	.7500000	.115484	.130115
55	.7500000	.408589	.452740
56	.7500000	.966576	.991066
57	.7500000	.129226	.035417

58	.7500000	.450106	.136649
59	.7500000	.990042	.471905
60	.7500000	.039434	.996843
61	.7500000	.151515	.012589
62	.7500000	.514232	.049723
63	.7500000	.999190	.189001
64	.7500000	.003238	.613120
65	.7500000	.012911	.948816
66	.7500000	.050976	.194258
67	.7500000	.193508	.626087
68	.7500000	.624252	.936409
69	.7500000	.938246	.238190
70	.7500000	.231761	.725821
71	.7500000	.712191	.796019
72	.7500000	.819899	.649491
73	.7500000	.590658	.910609
74	.7500000	.967125	.325600
75	.7500000	.127178	.878338
76	.7500000	.444014	.427440
77	.7500000	.987462	.978940
78	.7500000	.049522	.082465
79	.7500000	.188278	.302657
80	.7500000	.611319	.844223
81	.7500000	.950432	.526042
82	.7500000	.188442	.997287

83	.7500000	.611727	.010822
84	.7500000	.950068	.042818
85	.7500000	.189755	.163938
86	.7500000	.614991	.548250
87	.7500000	.947108	.990688
88	.7500000	.200378	.036901
89	.7500000	.640906	.142159
90	.7500000	.920582	.487798
91	.7500000	.292444	.999404
92	.7500000	.827682	.002381
93	.7500000	.570498	.009500
94	.7500000	.980120	.037638
95	.7500000	.077939	.144886
96	.7500000	.287457	.495576
97	.7500000	.819301	.999922
98	.7500000	.592186	.000313
99	.7500000	.966007	.001252
100	.7500000	.131350	.005003

The different solution trajectories of chaotic equations form patterns called *strange attractors*. If similar patterns appear in the strange attractor at different scales (larger or smaller, governed by some multiplier or scale factor **r,** as we saw previously), they are said to be *fractal.* They have a fractal dimension D, governed by the relationship $N = r^D$. Chaos equations like the one here (namely, the logistic equation) generate fractal patterns.

Why Chaos?

Why chaos? Does it have a physical or biological function? The answer is yes.

One role of chaos is the prevention of *entrainment*. In the old days, marching soldiers used to break step when marching over bridges, because the natural vibratory rate of the bridge might become entrained with the soldiers' steps, and the bridge would become increasingly unstable and collapse. (That is, the bridge would be destroyed due to bad vibes.) Chaos, by contrast, allows individual components to function somewhat independently.

A chaotic world economic system is desirable in itself. It prevents the development of an international business cycle, whereby many national economies enter downturns simultaneously. Otherwise national business cycles may become harmonized so that many economies go into recession at the same time. Macroeconomic policy co-ordination through G7 (G8, whatever) meetings, for example, risks the creation of economic entrainment, thereby making the world economy less robust to the absorption of shocks.

"A chaotic system with a strange attractor can actually dissipate disturbance much more rapidly. Such systems are highly initial-condition sensitive, so it might seem that they cannot dissipate disturbance at all. But if the system possesses a strange attractor which makes all the trajectories acceptable from the functional point of view, the initial-condition sensitivity provides the most effective mechanism for dissipating disturbance" [7].

In other words, because the system is so sensitive to initial conditions, the initial conditions quickly become unimportant, provided it is the strange attractor itself that delivers the benefits. Ary Goldberger of the Harvard Medical School has argued that a healthy heart is chaotic [8]. This comes from comparing electrocardiograms of normal individuals with heart-attack patients. The ECG's of healthy patients have complex irregularities, while those about to have a heart attack show much simpler rhythms.

How Fast Do Forecasts Go Wrong?—The Lyapunov Exponent

The *Lyapunov exponent* λ is a measure of the exponential rate of divergence of neighboring trajectories.

We saw that a small change in the initial conditions of the logistic equation (Table 1) resulted in widely divergent trajectories after a few iterations. How fast these trajectories diverge is a measure of our ability to forecast.

For a few iterations, the three trajectories of Table 1 look pretty much the same. This suggests that short-term prediction may be possible. A prediction of "x(n+1) = .75", based solely on the first trajectory, starting at x(0) = .75, will serve reasonably well for the other two trajectories also, at least for the first few iterations. But, by iteration 20, the values of x(n+1) are quite different among the three trajectories. This suggests that long-term prediction is impossible.

So let's think about the short term. How short is it? How fast do trajectories diverge due to small observational errors, small shocks, or other small differences? That's what the Lyapunov exponent tells us.

Let ε denote the error in our initial observation, or the difference in two initial conditions. In Table 1, it could represent the difference between .75 and .7499, or between .75 and .74999.

Let R be a distance (plus or minus) around a reference trajectory, and suppose we ask the question: how quickly does a second trajectory— which includes the error ε — get outside the range R? The answer is a function of the number of steps **n**, and the Lyapunov exponent λ , according to the following equation (where "exp" means the exponential **e** = 2.7182818…, the basis of the natural logarithms):

$$\mathbf{R} = \varepsilon \bullet \mathbf{exp}(\lambda\ \mathbf{n}).$$

For example, it can be shown that the Lyapunov exponent of the logistic equation is λ = log 2 = .693147 [9]. So in this instance, we have **R** = $\varepsilon \bullet$ exp(.693147 **n**).

So, let's do a sample calculation, and compare with the results we got in Table 1.

Sample Calculation Using a Lyapunov Exponent

In Table 1 we used starting values of .75, .7499, and .74999. Suppose we ask the question, how long (at what value of **n**) does it take us to get out of the range of +.01 or -.01 from our first (constant) trajectory of x(n) = .75? That is, with a slightly different starting value, how many steps does it take before the system departs from the interval (.74, .76)?

In this case the distance R = .01. For the second trajectory, with a starting value of .7499, the change in the initial condition is ε = .0001 (that is, ε = 75-.7499). Hence, applying the equation R = ε • exp(λ n), we have

.01 = .0001 exp (.693147 n).

Solving for n, we get n = 6.64. Looking at Table 1, we see that that for n = 7 (the 7th iteration), the value is x(7) = .762688, and that this is the first value that has gone outside the interval (.74, .76).

Similarly, for the third trajectory, with a starting value of .74999, the change in the initial condition is ε = .00001 (i.e., . ε = 75-.74999). Applying the equation R = ε • exp(λ n) yields

.01 = .00001 exp (.693147 n).

Which solves to n = 9.96. Looking at Table 1, we see that for n = 10 (the 10th iteration), we have x(10) = .739691, and this is the first value outside the interval (.74, .76) for this trajectory.

In this sample calculation, the system diverges because the Lyapunov exponent is positive. If it were the case the Lyapunov exponent were negative, λ < 0, then exp(λ n) would get smaller with each step. So it must be the case that λ > 0 for the system to be chaotic.

Note also that the particular logistic equation, x(n+1) = 4 x(n) [1-x(n)], which we used in Table 1, is a simple equation with only one variable, namely x(n). So it has only one Lyapunov exponent. In general, a system with **M** variables may have as many as **M** Lyapunov exponents. In that case, *an attractor is chaotic if at least one of its Lyapunov exponents is positive.*

The Lyapunov exponent for an equation *f* (x(n)) is the average absolute value of the natural logarithm (log) of its derivative:

$$\lambda = \sum_{n \to \infty} (1/n) \log |df/dx(n)|$$

For example, the derivative of the right-hand side of the logistic equation

x(n+1) = 4 x(n)[1-x(n)] = 4 x(n) – 4 x(n)^2

is

4 - 8 x(n) .

Thus for the first iteration of the second trajectory in Table 1, where x(n) = .7502, we have | d*f* /dx(n)| = | 4[1-2 (.7502)] | = 2.0016, and log (2.0016) = .6939. If we sum over this and subsequent values, and take the average, we have the Lyapunov exponent. In this case the first term is already close to the true value. But it doesn't matter. We can start with x(0) = .1, and obtain the Lyapunov exponent. This is done in Table 2, below, where after only ten iterations the empirically calculated Lyapunov exponent is .697226, near its true value of .693147.

Table 2: Empirical Calculation of Lyapunov Exponent from the Logistic Equation with x(0) = .1

x(n)	log|d*f*/dx(n)|

Iteration:

1	.360000	.113329
2	.921600	1.215743
3	.289014	.523479
4	.821939	.946049
5	.585421	-.380727
6	.970813	1.326148
7	.113339	1.129234
8	.401974	-.243079
9	.961563	1.306306
10	.147837	1.035782
Average		**.697226**

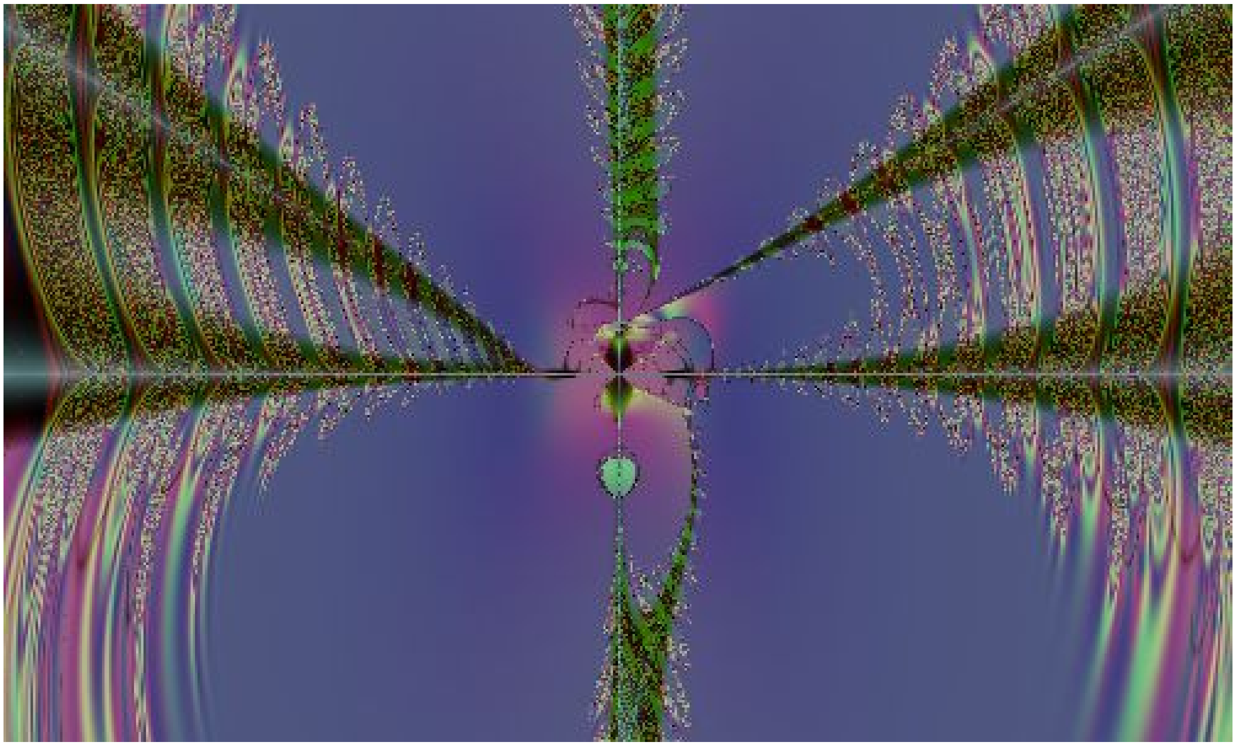
Enough for Now

In the next part of this series, we will discuss fractals some more, which will lead directly into economics and finance. In the meantime, here are some exercises for eager students.

Exercise 1: Iterate the following system: x(n+1) = 2 x(n) mod 1. [By "mod 1" is meant that only the fractional part of the result is kept. For example, 3.1416 mod 1 = .1416.] Is this system chaotic?

Exercise 2: Calculate the Lyapunov exponent for the system in Exercise 1. Suppose you change the initial starting point x(0) by .0001. Calculate, using the Lyapunov exponent, how many steps it takes for the new trajectory to diverge from the previous trajectory by an amount greater than .002.

Finally, here is a nice fractal graphic for you to enjoy:



Notes

[1] Eugene F. Fama, "Mandelbrot and the Stable Paretian Hypothesis," *Journal of Business*, 36, 420-429, 1963.

[2] If you really want to know why, read J. Aitchison and J.A.C. Brown, *The Lognormal Distribution*, Cambridge University Press, Cambridge, 1957.

[3] J. Orlin Grabbe, *Three Essays in International Finance*, Department of Economics, Harvard University, 1981.

[4] The Sierpinski Carpet graphic and the following one, the Lorentz attractor graphic, were taken from the web site of Clint Sprott: http://sprott.physics.wisc.edu/ .

[5] Ernest Lawrence Rossi, "Archetypes as Strange Attractors," *Psychological Perspectives*, 20(1), The C.G. Jung Institute of Los Angeles, Spring-Summer 1989.

[6] E. N. Lorenz, "Deterministic Non-periodic Flow," *J. Atmos. Sci.*, 20, 130-141, 1963.

[7] M. Conrad, "What is the Use of Chaos?", in Arun V. Holden, ed., *Chaos*, Princeton University Press, Princeton, NJ, 1986.

[8] Ary L. Goldberger, "Fractal Variability Versus Pathologic Periodicity: Complexity Loss and Stereotypy In Disease," *Perspectives in Biology and Medicine*, 40, 543-561, Summer 1997.

[9] Hans A. Lauwerier, "One-dimensional Iterative Maps," in Arun V. Holden, ed., *Chaos*, Princeton University Press, Princeton, NJ, 1986.

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from *The Laissez Faire City Times*, Vol 3, No 22, May 31, 1999

[Email Reply]

In Praise of Chaos

by J. Orlin Grabbe
Speech Presented to Eris Society, August 12, 1993

Introduction: The Intrusion of Eris

Chaos has a bad name in some parts. It was chaos that brought us the Trojan War (Robert Graves, *The Greek Myths*, chapter 159). Eris, goddess of chaos, upset at not being invited to the wedding of Peleus and Thetis, showed up anyway and rolled a golden apple marked "kalliste" ("for the prettiest one") among the guests. Each of the goddesses Hera, Athena, and Aphrodite claimed the golden apple as her own. Zeus, no fool, appointed Paris, son of Priam, king of Troy, judge of the beauty contest. Hermes brought the goddesses to the mountain Ida, where Paris first tried to divide the apple among the goddesses, then made them swear they wouldn't hold the decision against them. Hermes asked Paris if he needed the goddesses to undress to make his judgment, and he replied, Of course. Athena insisted Aphrodite remove her magic girdle, the sexy underwear that made everyone fall in love with her, and Aphrodite retorted Athena would have to remove her battle helmet, since she would look hideous without it.

As Paris examined the goddesses individually, Hera promised to make Paris the lord of Asia and the richest man alive, if she got the apple. Paris said he couldn't be bribed. Athena promised to make Paris victorious in all his battles, and the wisest man alive. Paris said there was peace in these parts. Aphrodite stood so close to Paris he blushed, and not only urged him not to miss a detail of her lovely body, but said also that he was the handsomest man she had seen lately, and he deserved a woman as beautiful as she was. Had he heard about Helen, the wife of the king of Sparta? The goddess promised Paris she would make Helen fall in love with him. Naturally Paris gave the apple to Aphrodite, and Hera and Athena went off fuming to plot the destruction of Troy. That is, Aphrodite got the apple, and Paris got screwed.

While the Greeks had a specific goddess dedicated to Chaos, early religions gave chaos an even more fundamental role. In the Babylonian New Year festival, Marduk separated Tiamat, the dragon of chaos, from the forces of law and order. This primal division is seen in all early religions. Yearly homage was paid to the threat of chaos's return. Traditional New Year festivals returned symbolically to primordial chaos through a deliberate disruption of civilized life. One shut down the temples, extinguished fires, had orgies and otherwise broke social norms. The dead mingled with the living; Afterward you purified yourself, reenacted the creation myth whereby the dragon of chaos was overthrown, and went back to normal. Everyone had fun, but afterward order was restored, and the implication was it was a good thing we had civilization, because otherwise people would always be putting out the fires and having orgies.

Around us in the world today we see the age-old battle between order and chaos. In the **international sphere**, the old order of communism has collapsed. In its place is a chaotic matrix of competing, breakaway states, wanting not only political freedom and at least a semi-market economy, but also their own money supplies and nuclear weapons, and in some cases a society with a single race, religion, or culture. Is this alarming or reassuring? We also have proclamations of a New World Order, on one hand, accompanied by the outbreak of sporadic wars and US bombing raids in Africa, Europe, and Asia, on the other.

In the **domestic sphere** we have grass roots political movements, such as the populist followers of H. Ross Perot challenging the old order imposed by the single-party Democratic- Republican monolith. We have a President who is making a mockery out of the office, and a Vice President who tells us we should not listen to any dissenting opinions with respect to global warming. Is this reassuring or alarming?

In the **corporate-statist** world of Japan we see the current demolition of the mythic pillars of Japanese society: the myth of high-growth, the myth of endless trust between the US and Japan, the myth of full employment, the myth that land and stock prices will always rise, and the myth that the Liberal Democratic Party will always remain in power. Is the shattering of these myths reassuring or alarming?

In fact, wherever we look, central command is losing control. Even in the sphere of the human mind we have increasing attention paid to cases of multiple personality. The most recent theories see human identity and the human ego as a network of cooperative subsystems, rather than a single entity. (Examples of viewpoint are found in Robert Ornstein, *Multimind*, and Michael Gazzanaga, *The Social Brain*.) If, as Carl Jung claimed, "our true religion is a monotheism of consciousness, a possession by it, coupled with a fanatical denial of the existence of fragmentary autonomous systems," then it can be said that psychological polytheism is on the rise. Or, as some would say, mental chaos. Is this reassuring or alarming?

Myth of Causality Denies Role of Eris

The average person, educated or not, is not comfortable with chaos. Faced with chaos, people begin talking about the fall of Rome, the end of time. Faced with chaos people begin to **deny** its existence, and present the alternative explanation that what appears as chaos is a hidden agenda of historical or prophetic forces that lie behind the apparent disorder. They begin talking about the "laws of history" or proclaiming that "God has a hidden plan". The creation, Genesis, was preceded by chaos (*tohu-va- bohu*), and the New World Order (the millennium), it is claimed, will be preceded by pre-ordained apocalyptic chaos. In this view of things, chaos is just part of a master agenda. Well, is it really the case there is a hidden plan, or does the goddess Eris have a non- hidden non-plan? Will there be a Thousand Year Reign of the Messiah, or the Thousand Year Reich of Adolph Hitler, or are these one and the same?

People are so uncomfortable with chaos, in fact, that Newtonian science as interpreted by Laplace and others saw the underlying reality of the world as **deterministic**. If you knew the initial conditions you could predict the future far in advance. With a steady hand and the right cue tip, you could run the table in pool. Then came quantum mechanics, with uncertainty and indeterminism, which even Einstein refused to accept, saying "God doesn't play dice." Philosophically, Einstein couldn't believe in a universe with a sense of whimsy. He was afraid of the threatened return of chaos, preferring to believe **for every effect there was a cause**. A consequence of this was the notion that if you could control the cause, you could control the effect.

The modern proponents of law and order don't stop with the assertion that for every effect, there is a cause. And they also assert they **"know" the cause**. We see this attitude reflected by social problem solvers, who proclaimed: "*The cause of famine in Ethiopia is lack of food in Ethiopia*." So we had rock crusades to feed the starving Ethiopians and ignored the role of the Ethiopian government. Other asserted: "*The case of drug abuse is the presence of drugs*," so they enacted a war on certain drugs which drove up their price, drove up the profit margins available to those who dealt in prohibited drugs, and created a criminal subclass who benefited from the prohibition. Psychologists assert: "*The reason this person is this way is because such-and-such happened in childhood, with parents, or siblings, or whatever*." So any evidence of abuse, trauma, or childhood molestation--which over time should assume a trivial role in one's life--are given infinite power by the financial needs of the psychotherapy business.

You may respond: "Well, but these were just misidentified causes; there really is a cause." Maybe so, and maybe not. Whatever story you tell yourself, you can't escape the fact that to you personally "the future is a blinding mirage" (Stephen Vizinczey, *The Rules of Chaos*). You can't see the future precisely because you don't really know what's causing it. The myth of causality denies the role of Eris. Science eventually had to acknowledge the demon of serendipity, but not everyone is happy with that fact. The political world, in the cause-and-effect marketing and sales profession, has a vested interest in denying its existence.

Approaches to Chaos

In philosophy or religion there are three principal schools of thought (in a classification I'll use here). Each school is distinguished by its basic philosophical outlook on life. The First School sees the universe as indifferent to humanity's joys or sufferings, and accepts chaos as a principle of restoring balance. The Second School sees humanity as burdened down with suffering, guilt, desire, and sin, and equates chaos with punishment or broken law. The Third School considers chaos an integral part of creativity, freedom, and growth.

I. First School Approach: Attempts to Impose Order Lead to Greater Disorder

Too much law and order brings its opposite. Attempts to create World Government will lead to total anarchy. What are some possible examples?

- The Branch Davidians at Waco. David Koresh's principal problem was, according to one FBI spokesman, that he was "thumbing his nose at the law". So, to preserve order, the forces of law and order brought chaos and destruction, and destroyed everything and everyone. To prevent the misuse of firearms by cult members, firearms were marshaled to randomly kill them. To prevent alleged child abuse, the forces of law and order burned the children to death.
- Handing out free food in "refugee" camps in Somalia leads to greater number of starving refugees, because the existence of free food attracts a greater number of nomads to the camps, who then become dependent on free food, and starve when they are not fed.
- States in the US. are concerned about wealth distribution. But, to finance themselves, more and more states have turned to the lottery. These states thereby create inequality of wealth distribution by giving away to a few, vast sums of cash extracted from the many.

The precepts of the first school find expression in a number of Oriental philosophies. In the view of this school, what happens in the universe is a fact, and does not merit the labels of "good" or "bad", or human reactions of sympathy or hatred. Effort to control or alter the course of macro events (as opposed to events in ones personal life) is wasted. One should cultivate detachment and contemplation, and learn elasticity, learn to go with the universal flow of events. This flow tends toward a balance. This view finds expression in the Tao Teh King:

> The more prohibitions you have,
> the less virtuous people will be
> The more weapons you have,
> the less secure people will be.
> The more subsidies you have,
> the less self-reliant people will be.
>
> Therefore the Master says:
> I let go of the law,
> and people become honest.
> I let go of economics,
> and people become prosperous.
> I let go of religion,
> and people become serene.
> I let go of all desire for the common good,
> and the good becomes common as grass.
> (Chapter 57, Stephen Mitchell translation.)

You don't fight chaos any more than you fight evil. "Give evil nothing to oppose, and it will disappear by itself" (Tao Teh King, Chapter 60). Or as Jack Kerouac said in *Dr. Sax*: "The universe disposes of its own evil." Again the reason is a principal of balance: You are controlled by what you love and what you hate. But hate is the stronger emotion. Those who fight evil necessarily take on the characteristics of the enemy and become evil themselves. Organized sin and organized sin-fighting are two sides of the same corporate coin.

II. Second School Approach: Chaos is a Result of Breaking Laws

In the broadest sense, this approach a) **asserts society is defective**, and then b) **tells us the reason it's bad is because we've done wrong by our lawless actions**. This is the view often presented by the front page of any major newspaper. It's a fundamental belief in Western civilization.

In early Judaism and fundamentalist Christianity, evil is everywhere and it must be resisted. There is no joy or pleasure without its hidden bad side. God is usually angry and has to be propitiated by sacrifice and blood. The days of Noah ended in a flood. Sodom and Gomorra got atomized. Now, today, it's the End Time and the wickedness of the earth will be smitten with the sword of Jesus or some other Messiah whose return is imminent.

In this context, chaos is punishment from heaven. Or chaos is a natural degenerate tendency which must be alertly resisted. In the Old Testament Book of Judges, a work of propaganda for the monarchy, it is stated more than once: "In those days there was no king in Israel: every man did that which was right in his own eyes" (Judg. 17:6; 21:25). Doing what appeals to you was not considered a good idea, because, as Jeremiah reminds us "The heart [of man] is deceitful above all things and desperately wicked" (Jer. 17:9).

And in the New Testament, the rabbinical lawyer Paul says "by the law is the knowledge of sin" (Rom. 3:20), and elsewhere is written, "Whososever committeth sin transgresseth also the law: for sin is the transgression of the law." (1 John 3:4). And, naturally, "the wages of sin is death" (Rom. 6:23).

New age views of karma are similar. If you are bad, as somehow defined, you built up bad karma (New Age view), or else God later burns you with fire (fundamentalist Christian view). For good deeds, you get good karma or treasures in heaven. It's basically an **accountant's view of the world**. Someone's keeping a balance sheet of all your actions, and toting up debits or credits. Of course, some religions allow you to wipe the slate clean in one fell swoop, say by baptism, or an act of contrition, which is sort of like declaring bankruptcy and getting relief from all your creditors. But that's only allowed because there has already been a blood sacrifice in your place. Jesus or Mithra or one of the other Saviors has already paid the price. But even so, old Santa Claus is up there somewhere checking who's naughty or nice.

What is fundamental about this approach is not the specific solution to sin, or approach to salvation, but the general pessimistic outlook on the ordinary flow of life. The first Noble Truth of Buddha was that "Life is Sorrow". In the view of Schopenhauer, Life is Evil, and he says "Every great pain, whether physical or spiritual, declares what we deserve; for it could not come to us if we did not deserve it" (*The World as Will and Representation*). Also in the Second School bin of philosophy can be added Freud, with his Death Wish and the image of the unconscious as a murky swamp of monsters. Psychiatry in some interpretations sees the fearful dragon of chaos, Tiamat, lurking down below the civilized veneer of the human cortex.

The liberal's preoccupation with social "problems" and the Club of Rome's obsession with entropy are essentially expressions of the Second School view. **Change, the fundamental motion of the universe, is bad**. If a business goes broke, it's never viewed as a source of creativity, freeing up resources and bringing about necessary changes. It's just more unemployment. The unemployment-inflation tradeoff as seen by Sixties Keynesian macroeconomics is in the Second School spirit. These endemic evils must be propitiated by the watchful Priests of Fiscal Policy and the Federal Reserve, and you can only reduce one by increasing the other. This view refuses to acknowledge that one of the positive roles of the Market is as a job destroyer as well as a job creator.

More generally, the second school has generated whole industries of **"problem solvers"**-- politicians, bureaucrats, demagogues, counselors, and charity workers who have found the way to power, fame, and wealth lies in championing causes and mucking about in other people's lives. Whatever their motivations, they operate as parasites and vampires who are healthy only when others are sick, whose well-being increases in direct proportion to other people's misery, and whose method of operation is to give the appearance of working on the problems of others. Of course if the problems they champion were actually solved, they would be out of a job. Hence they are really interested in the process of "solving" problems--not in actual solutions. They create chaos and destruction under the pretense of chaos control and elimination.

III. Third School Approach: Chaos is Necessary for Creativity, Freedom, and Growth

You find this view in a few of the ancient Greek writers, and more recently in Nietzsche. Nietzsche says: "One must still have chaos in one to give birth to a dancing star." The first fundamental point of view here is: **Existence is pure joy**. If you don't see that, your perception is wrong. And we are not talking about Mary Baker Eddy Christian Science denial of the facts. In this approach you are supposed to learn to alchemically transmute sorrow into joy, chaos into art. You exult in the random give and take of the hard knocks of life. It's a daily feast. Every phenomenon is an Act of Love. Every experience, however serendipitous, is necessary, is a sacrament, is a means of growth.

> "Saying Yes to life even in its strangest and hardest problems, the will to life rejoicing over its own inexhaustibility even in the very sacrifice of its highest types--that is what I called Dionysian, that is what I guessed to be the bridge to the psychology of the tragic poet. Not in order to be liberated from terror and pity, not in order to purge oneself of a dangerous affect by its vehement discharge-- Aristotle understood it that way [as do the Freudians who think one deals with ones neuroses through one's art, a point of view which Nietzsche is here explicitly rejecting]--but in order to be oneself the eternal order of becoming, beyond all terror and pity--that joy which included even joy in destroying." (*Twilight of the Idols*).

It is an approach centered in the here and now. You cannot foresee the future, so you must look at the present. But because "nothing is certain, nothing is impossible" (*Rules of Chaos*). You are free and nobody belongs to you. In the opening paragraphs of *Tropic of Cancer*, Henry Miller says: "It is now the fall of my second year in Paris. I was sent here for a reason I have not yet been able to fathom. I have no money, no resources, no hopes. I am the happiest man alive."

Your first responsibility is to take care of yourself, so you won't be a burden to other people. If you don't do at least that, how can you be so arrogant as to think you can help others? You make progress by adapting to your own nature. In Rabelais' *Gargantua* the Abbey of Theleme had the motto: *Fay ce que vouldras*, or "Do as you will." Rabelais (unlike the Book of Judges) treats this in a very positive light. The implication is: Don't go seeking after some ideal far removed from your own needs. Don't get involved in some crusade to save the human race--because you falsely think that is the noble thing to do--when what you may really want to do, if you are honest with yourself, is to stay home, grow vegetables, and sell them in a roadside market. (Growing vegetables is, after all, real growth--more so than some New Age conceptions.) You have no obligation under the sun other than to discover your real needs, to fulfill them, and to rejoice in doing so.

In this approach you give other people the right to make their own choices, but you also hold them responsible for the consequences. **Most social "problems", after all, are a function of the choices people make, and are therefore insolvable in principle, except by coercion**. One is not under any obligation to make up for the effects of other people's decisions. If, for example, people (poor or rich, educated or not) have children they can't care for or feed, one has no responsibility to make up for their negligence or to take on one's own shoulders responsibility for the consequent suffering. You can, if you wish, if you want to become a martyr. If you are looking to become a martyr, the world will gladly oblige, and then calmly carry on as before, the "problems" unaltered.

One may, of course, choose to help the rest of the world to the extent that one is able, assuming one knows how. But it is a *choice*, not an obligation. Modern political correctness and prostituted religion have tried to turn all of what used to be considered *virtues* into social *obligations*. Not that anyone is expected to really practice what they preach; rather it is intended they feel guilty for not doing so, and once the guilt trip is underway, their behavior can be manipulated for political purposes.

What would, after all, be left for social workers to do if all social problems were solved? One would still need challenges, so presumably people would devote themselves to creative and artistic tasks. One would still need chaos. One would still need Eris rolling golden apples.

Conclusion

In the revelation given to Greg Hill and Kerry Thornley, authors of **Principia Discordia**, or How I Found Goddess and What I Did to Her When I Found Her, the goddess Eris (Roman Discordia) says: "I am chaos. I am the substance from which your artists and scientists build rhythms. I am the spirit with which your children and clowns laugh in happy anarchy. I am chaos. I am alive, and I tell you that you are free."

Today, in Aspen, Eris says: *I am chaos. I am alive, and I tell you that you are free*.

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. . . inspecting the global underbelly: privacy, money laundering, espionage.

"What forbids us to tell the truth, laughingly?"--Horace, *Satires*, I.24

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The Hat Color Problem

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End of Ordinary Money, Part 1	End of Ordinary Money, Part 2
NSA, Crypto AG, and the Iran-Iraq War	Crypto AG: NSA's Trojan Whore?
Charles Hayes: A Prison Interview	Is the FBI Railroading Charles Hayes?
Michael Riconosciuto on Encryption	ECHELON: Global Surveillance

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April 9, 2001

The Laissez Faire City Times

In the Current Issue:
• **A Derivative Life, by J. Orlin Grabbe**
• **Zoned Out, by Russell Madden**
• **Bush and the GOP: They're All Socialists Now, by Michael Chapman**
• **Make-Believe Campaign Finance Reform, by Tibor Machan**
• **Panic Attack: the Green Scare, Part 2, by Jim Peron**

Recent News Links

	Note: the most recent links are below, at the end of this table. The selected links here are temporary, and will slowly disappear.
Jack Sarfatti	**Get your mind out of the political gutter and think about the future.**
Curse of Lago Agrio	**Drug War threatens havoc in South America.**
NSA Backdoors	**Why you shouldn't buy software made in the USA.**
Russia's Road to Corruption	**How the Clinton administration exported government instead of free enterprise.**
Linux Crypto	**Guide to Linux encryption.**
State Surveillance in the Internet	**The spectrum of suspicion.**
Rijndael	**The Advanced Encryption Standard final choice.**
Jerusalem	**The Red Heifer wants you to blow it up.**
Clone Jesus	**This is my body, my blood, my DNA sample. Now we just need a virgin.**

Electronic Cash	**Survey by Big Brother's Financial Crimes Enforcement Network (FinCEN).**
Bank Fraud	**Who carries the risk of Internet bank fraud?**
Turing's Enigma	**Alan Turing's Treatise on the Enigma Machine**
Encrypted Voice	**Encrypted voice communication began the digital age.**
Dromology	**The aethetics of war and deception.**
QAZ	**The worm that ate Microsoft.**
FinCEN Strategic Plan 2000-2005	**How the financial spies plan to keep busy the next 5 years.**
Carnivore	**"Independent" Technical Review completed.**
Break PGP	**Beware of keystroke monitors like this one.**
Second Amendment	**Wiesenthal Nazis smear women's gun group.**
Drug War	**Mexico's Secretary of State says the drug trade should be legalized.**
Holocaust Industry Shakedown	**How to make big bucks off the dead.**
Astalavista Group	**The official site of underground information.**
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Cracking & Reversing	• **http://tsehp.cjb.net** • **http://www.woodmann.com/fravia/** • **http://go.to/tsehp** • **http://www.searchlores.org** • **http://fravia.2113.ch/index.htm** • **http://qu00l.net/s/www.searchlores.org/**
FBI Passphrase Theft	**Court order to seize encryption keys & passphrases via a keystroke logger (.pdf file).**
Morlocks	**Underground bases and tunnels.**
The Wilcher Report	**Learn why the Branch Davidians were assassinated and the evidence burned. (Huge .pdf file.)**

Chinese Spying Guide	How the Chinese collect foreign technological information.
Video Retrival	The technology of spook spy cams.
Voices in Your Head	"By proper choice of pulse characteristics, intelligible speech may be created." Such as: "I am your God. I command you to kill the enemies of the government."
Inside the NSA	What astronomers found when they took over an NSA satellite listening post.
PROMIS & the RCMP	*Insight* on the Inslaw PROMIS Scandal.
Underground	Enter the world of hacking.
Riconosciuto & Videnieks	PROMIS Trail Leads to Justice
Steganography	A Tutorial.
Hiding URLs	How to Obscure Any URL: a lesson of interest to spammers, terrorists, child pornographers, and the scum-sucking FBI.
Privacy Not	Serious Flaw in OpenPGP
Military Plunder	How to steal millions from the US Defense Department.
Cuba de mi Amor	The Cuban Submarine Crisis
SS Attacks Privacy	Organized Crime Raids Gold-Age

All is not lost—the unconquerable will,
And study of revenge, immortal hate,
And courage never to submit or yield.
—John Milton, *Paradise Lost*

Today's News Articles

Mathematics

Why Mathematicians Care About Hat Color

Hamming codes.

By SARA ROBINSON

BERKELEY, Calif., April 9 — It takes a particularly clever puzzle to stump a mind accustomed to performing mental gymnastics.

So it's no ordinary puzzle that's spreading through networks of mathematicians like a juicy bit of gossip. Known as "the hat problem" in its most popular incarnation, this seemingly simple puzzle is consuming brain cycles at universities and research labs across the country and has become a vibrant topic of discussion on the Internet.

The reason this problem is so captivating, mathematicians say, is that it is not just a recreational puzzle to be solved and put away.

Rather, it has deep and unexpected connections to coding theory, an active area of mathematical research with broad applications in telecommunications and computer science.

In their attempts to devise a complete solution to the problem, researchers are proving new theorems in coding theory that may have applications well beyond mathematical puzzles.

"This puzzle is unique since it connects to unsolved mathematical questions," said Dr. Joe Buhler, deputy director of the Mathematical Sciences Research Institute here and a hat problem enthusiast.

The hat problem goes like this:

Three players enter a room and a red or blue hat is placed on each person's head. The color of each hat is determined by a coin toss, with the outcome of one coin toss having no effect on the others. Each person can see the other players' hats but not his own.

No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, the players must simultaneously guess the color of their own hats or pass. The group shares a hypothetical $3 million prize if at least one player guesses correctly and no players guess incorrectly.

The same game can be played with any number of players. The general problem is to find a strategy for the group that maximizes its chances of winning the prize.

One obvious strategy for the players, for instance, would be for one player to always guess "red" while the other players pass. This would give the group a 50 percent chance of winning the prize. Can the group do better?

Most mathematicians initially think not. Since each person's hat color is independent of the other players' colors and no communication is allowed, it seems impossible for the players to learn anything just by looking at one another. All the players can do, it seems, is guess.

"I tell someone the problem and they think they don't have the conditions right," said Dr. Peter Winkler, director of fundamental mathematics research at Bell Labs of Lucent Technologies in Murray Hill, N.J. "But if you try to prove it's impossible, it doesn't quite work."

Mathematicians credit the problem to Dr. Todd Ebert, a computer science instructor at the University of California at Irvine, who introduced it in his Ph.D. thesis at the University of California at Santa Barbara in 1998.

Dr. Ebert said he discovered the problem's appeal only recently, when he offered extra credit to his students for solving a seven-player version he called the "seven prisoners puzzle."

Next thing he knew, the problem was posted on Internet news groups and in chat rooms. "I started getting e-mail from all over the country," Dr. Ebert said.

Meanwhile, Dr. Winkler, a well- known collector and distributor of clever puzzles, heard the problem from a colleague and spread it widely. It has cropped up at Microsoft Research in Redmond, Wash., at Hewlett-Packard Laboratories in Palo Alto, Calif., and at mathematics, statistics and computer science departments at

universities across the country.

The problem has even spread to the Caribbean. At a workshop at a research institute in Barbados, one hardy group of theoretical computer scientists stayed up late one rum- soaked night, playing a drinking game based on the puzzle.

It spread to Berkeley after Dr. Winkler bumped into Dr. Elwyn Berlekamp, a professor in the Berkeley math department, at a conference in New Orleans in January.

"I told him about the problem and next thing I knew he was leaving messages on my hotel phone saying, `Great problem, haven't gotten it yet,' then finally, `I got it,' " Dr. Winkler said. "I thought, with his knowledge of coding theory, he'd find that approach, and he didn't disappoint me."

Dr. Berlekamp, a coding theory expert, said he figured out the solution to the simplest case in about half an hour, but he saw the coding theory connection only while he was falling asleep that night.

"If you look at old things that you know from a different angle, sometimes you can't see them," he said.

The first thing Dr. Berlekamp saw was that in the three-player case, it is possible for the group to win three- fourths of the time.

Three-fourths of the time, two of the players will have hats of the same color and the third player's hat will be the opposite color. The group can win every time this happens by using the following strategy: Once the game starts, each player looks at the other two players' hats. If the two hats are different colors, he passes. If they are the same color, the player guesses his own hat is the opposite color.

This way, every time the hat colors are distributed two and one, one player will guess correctly and the others will pass, and the group will win the game. When all the hats are the same color, however, all three players will guess incorrectly and the group will lose.

"If you look at the total number of guesses made, it's still the case that half are right and half wrong," Dr. Winkler said. "You only make progress if, when players are guessing wrong, a great many are guessing wrong."

The strategy gets far more complicated for larger numbers of players.

Still, it all comes down to making sure that most of the time no one is wrong and occasionally everyone is wrong at once.

As it turns out, this requirement can be perfectly met only when the number of players is one less than a power of two (three, seven, 15 and so on.)

For example, in the game with 15 players, there is a strategy for which the group is victorious 15 out of every 16 times they play.

This strategy can be described using elegant mathematical structures known as Hamming codes. Hamming codes, named after Richard Hamming, the mathematician who discovered them, are basic tools studied by engineering students all over the world.

Hamming codes straddle the boundary between two types of mathematical objects: error correcting codes and covering codes.

Error correcting codes, techniques for correcting errors in data sent across noisy channels, are used in everything from cell phones to compact discs. Covering codes can be used to compress data so they take up less space in a computer's memory.

"Hamming codes are perfect structures, a lot like crystals, where you can't move an

atom in them or they are completely destroyed," said Dr. Amin Shokrollahi, chief scientist at Digital Fountain, which uses coding theory to speed up Internet data transmissions. "When you take the hat problem apart and look at its core, you see what you need are exactly Hamming codes."

When the game is played with fewer than nine players, the optimal solution can be determined using various types of codes. For larger numbers that aren't one less than a power of two, a strategy designed around the Hamming code solution works closer and closer to 100 percent of the time as the number of players grows.

Dr. Hendrik Lenstra, a professor of mathematics at Berkeley, and Dr. Gadiel Seroussi, director of information theory research at HP Labs, have developed a new type of covering code to define an even better strategy for large numbers of players.

While their strategy is the best so far, they don't know that it is always optimal. The optimal solution to the hat problem, for all numbers of players, is still unknown.

"We're still working on it," Dr. Seroussi said. "And as a consequence of working on this problem, we've got some results in coding theory that are interesting in and of themselves."

For now, researchers say, it seems unlikely that a solution will have immediate practical applications. Still, one never knows what the future might hold. "My experience is that any mathematics I've done is useful eventually," Dr. Seroussi said.

Practically useful or not, for some researchers the hat problem has interesting social implications. "I like problems that have philosophical punch lines," Dr. Berlekamp said, citing two life lessons that can be gleaned from the puzzle:

"The first is that it's O.K. to be wrong as long as you contrive not to be wrong alone," he said. "The other, more important lesson is a need for teamwork that goes against the grain of most mathematicians. If the evidence suggests someone on your team knows more than you, you should keep your mouth shut.

"Most of us assume that each player's strategy is oriented toward him getting it right, and it's not. It's the whole team."

***The New York Times*, April 10, 2001**

Recent Articles by J. Orlin Grabbe

The End of Ordinary Money

A series explaining the use of the monetary system for government surveillance, and why the growth of the information superspyway will increasingly lead to a system of anonymous digital cash. "The End of Ordinary Money, Part 1" appeared in *Liberty* (July 1995) with the title "The End of Ordinary Money." Part 2 was published in *Liberty* (November 1995) under the title "The Money Laundromat."

The End of Ordinary Money, Part 1	Read this if you want to know what this web page is about.
The End of Ordinary Money, Part 2	And this one also.
How to Launder Money	Learning the basic skills.
How to Launder Money in the Futures Market	How to give Hillary a bribe via cattle futures.
Another Whack Against GAK	The government's grip on cryptology slips.
Plot to Spy on Banks Outlined in White House Email	Memo from 1986 details NSC/DOJ plot to spy on banks.
Changing the Currency	Is it counterfeit, or is it just foreign?
History of the Euro	The single European currency.
Clipper Chip Banking	Sandia National Laboratories create non-anonymous "anonymity."
Money Laundering Through Casinos	The Nevada advantage.
Money Laundering Through CHIPS and SWIFT	Wholesale cleansing.
Money Laundering: The BCCI Affair: Senate Report	The Bank of Criminals and Covert-Agents International knew a thing or two about laundering.

- ***Information Technologies for Control of Money Laundering*:**
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  - Chapter 1: Electronic Money Laundering (pdf)
  - Chapter 2: The Mechanisms of Wire Transfer (pdf)
  - Chapter 3: Money Laundering and Law Enforcement (pdf)
  - Chapter 4: Technologies for Detecting Money Laundering (pdf)
  - Chapter 5: Privacy and Confidentiality (pdf)
  - Chapter 6: International Issues (pdf)

  - Chapter 7: Conclusions and Policy Options (pdf)
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- Money Laundering: A Framework for Understanding US Efforts Overseas (pdf)
- Money Laundering: FINCEN's Law Enforcement Role is Evolving (pdf)
- Money Laundering: Regulatory Oversight of Private Offshore Banking Activities (pdf)
- Private Banking: Raul Salinas, Citibank, and Alleged Money Laundering (pdf)

But wait . . . There's more below!

- **Digital Cash: see the articles in "News Behind the News"**
- **Cryptology: see the articles on cryptology in "News Behind the News"**

How to Investigate the Investigators

Two articles explaining how to do your own financial investigations of politicians, government officials, judges, and law enforcement people. The first part, a motivational piece, is my introduction to a Jim Norman article explaining the efforts of a group of Fifth Column hackers who do just that. I don't agree with everything in Norman's article, but it's a good intro to the second article, which explains the basics of detecting unexplained sources of wealth (which often as not may be bribes, kickbacks, or payoffs from the Cali cartel). The second article appeared (without my permission) in *Media Bypass* (March 1996) under the title "Hackers vs. Politicians."

- Hackers vs. Politicians, Part I
- Hackers vs. Politicians, Part II
- **OTHER CITIZEN INVESTIGATIONS**

America's Dreyfus Affair: The Case of the Death of Vince Foster	Parts 1-6. Essay by David Martin.
Waco Holocaust Electronic Museum	An investigation put together by an inquisitive citizen, Carol Valentine, without benefit of subpoenas and wiretaps. After you visit the Museum, reflect on the shabby "investigation" conducted by the House of Representatives in 1995.
The John Doe Times	Info on the Oklahoma City bombing.
Softwar	Charles R. Smith follows the money trail of U.S. crypto policy.
John Young Architect's Cryptome	Document collection on privacy, cryptology, military technology.
Interim Report on the Crash of TWA Flight 800 and the Actions of the NTSB and the FBI	The report of Commander William S. Donaldson, USN Ret. in cooperation with Associated Retired Aviation Professionals.

International Financial Markets, 3rd Edition

Click here to see table of contents, reviews, and ordering information.

The News Behind the News

Encroaching Tyranny & the Technology of Freedom

This section includes many articles on the news I wrote beginning in 1996 after my Vince Foster series ran into too many national security information blackouts for satisfactory completion. Most of the articles involve one or another of the principal topics introduced in "The End of Ordinary Money." The unifying theme is individual freedom and privacy in a world of government malfeasance, government surveillance, and increasing tyranny.

In addition to my own articles, I have collected a lot of hard-to-find material written by others. Within each news section, the articles are mostly listed chronologically.

- **MEDIA**

The Internet and the Death of the News Monopoly	Life without a priestly caste of editors.
60 Minutes Attacks Area 51!	Leslie Stahl comes to Reno, while Mike Wallace fondles his cash.
How Much is Sidney Blumenthal's Reputation Worth?	Calculation of the Drudge Report damage to a sleazy smear artist.

- **SPOOKS**

Is the FBI Railroading Charles Hayes?	The answer is Yes.
Affidavit of Robert M. Hayes	The U.S. Government plots to bomb its own consulate to "simulate terrorism".
Michael Riconosciuto on Encryption	Comments from architect of the PROMIS backdoor.
Who is Leo Wanta?	The collapse of the Russian ruble. Bill Clinton's short-term notes.
CIA Proprietary Companies?	Cash stashed for George Bush, Bill Casey, Vince Foster, Bill Clinton--and others living and dead?
Big Sky Cartel Zero's In	Politicians and the FBI involved in Montana drugs.
Interview with Chip Tatum	Terry Nelson, Montana Drugs, Chuck Hayes, etc.
The Tatum Chronicles	Perhaps the only accurate account of the drug & arms "Enterprise" of North, Clinton & Bush.
Child Sex in Montana	More northern attractions.
Charles Hayes: A Prison Interview	The future of politics.
The Last Circle	Info on the drug trade, from the files of Michael Riconosciuto (whose role is misinterpreted).
The Case Against Jonathan Pollard	Seymour Hersh examines the rape of U.S. security.
Jonathan Pollard Was No Jewish Patriot	Eric Margolis explains that wishing doesn't make it so.
The Sins of Jonathan Pollard	Pollard stole NSA 10-vol radio signals information manual.
The Redacted Cox Report	All material implicating Clinton was classified by White House.

Nuclear Lab Security	The report of the President's Foreign Intelligence Advisory Board.
Strategic Intelligence	Spooky stuff.
Comment on PROMIS article	Clueless con man discovers the past.

- **DIGITAL CASH**

A Guide to Digital Cash Articles on My Webpage	Read this first for background.
Digital Cash and the Future of Money	Appeared as "Introduction to Digital Cash" in *Liberty* (July 1998) and in the SIRS Renaissance data base (1999).
Concepts in Digital Cash	Basic concepts necessary to get started.
Digital Cash and the Regulators	We gonna regulate, just as soon as we figure out what it is.
The Mathematical Ideas Behind Digital Cash	Speech to Libertarian Party of Colorado.
Internet Payment Schemes, Part 1	Secure credit card systems. *i*KP, SET, EMV, CyberCash, First Virtual, FBOI, BankNet, Open Financial Exchange.
Internet Payment Schemes, Part 2	Adventures in Prague. Debt-Credit systems. NetCheque, Netbill, and CyberCoin.
Internet Payment Schemes, Part 3	Digital cash systems. NetCash, Mondex, & Digicash ecash.
Cryptography and Number Theory for Digital Cash	A tutorial.
Stefan Brands' System of Digital Cash	A digital cash system with real privacy. Consistency between the on-line & off-line systems.
Digital Signatures Illustrated	A tutorial.
Smart Cards and Private Currencies	The technology of Hayek's dream.

- **FIN DE SIECLE**

October Country
The Year 2000 Problem in Perspective
Financial Regulators Year 2000 Resources

- **POST-DIGITAL PRIVACY**

The End of Ordinary Money, Part 1	How Big Brother took over the financial system.
The End of Ordinary Money, Part 2	Your banker is a snitch. Money laundering law & digital cash.
Social Security Numbers as Identification Tokens	Protect yourself from government and private databases.
The International PGP Home Page	Encryption for the masses. Get your free PGP here.
Russian PGP Homepage	
PGP Faq: Diffie Hellman vs. RSA	Forget RSA. Use Diffie-Hellman (which is really El Gamal).
Transferring Defense Technology to Law Enforcement	Voice recognition and other matters.
Communications Privacy in a Digital Age	The costs of monitoring have dropped. We need to take steps to increase them.
Department of Justice Guidelines for Seizing Computers	[*Coming soon.*]

The World Financial Police Attack Anonymity	We want to know where your money is and what you are doing with it.
Technologies of Political Control	How governments keep the sheep obediently in line.
ECHELON: Global Surveillance	A global agreement, centered at the NSA, to spy on citizens.
Trust in Cyberspace	The National Academy of Sciences examines network security.
The Hidden Hand of ILETS	More on international tapping via "back doors" and ENFOPOL.
Interception Capabilities 2000	European Parliament report on Echelon and Communications Spying.
Microsoft's "NSA Backdoor"	An NSA backdoor? Probably not, but Microsoft "security" is clearly a myth.
National Security Agency Declassified	Document introduction by Jeffrey T. Richelson.
National Security Agency Documents	With commentary by intelligence author Jeffrey T. Richelson.

- **CRYPTOLOGY**

Security Risks in Government-Mandated "Key Recovery"	Nightmare on Clipper Street, part III.
Encryption, China, RSA, and Banking	Following the money code.
Michael Riconosciuto on Encryption	Comments from architect of the PROMIS backdoor.
Cryptography and Number Theory for Digital Cash	A tutorial.
How Secure is America's Nuclear Arsenal?	Public key cryptography and nuclear weapons.
NSA, Crypto AG, and the Iraq-Iran War	How the NSA bugged international cryptography software.
Letter from Bill Payne Regarding Cryptography at Sandia	
Crypto AG: The NSA's Trojan Whore?	More on how the NSA bugged international cryptography software.
Digital Signatures Illustrated	A tutorial.
The DES Algorithm Illustrated	A tutorial.
Black and White Test of Cryptographic Algorithms	William H. Payne explains how to choose an algorithm.
The Wassenaar Invasion of Privacy	Global agreement on cryptography restriction.
International Cryptography Freedom	Protect the world from Wassenaar.
Crack of GSM Cell Phones	
Crack of Microsoft's PPTP Protocol	Virtual Un-Private Networks.
Deep Crack: The EFF Builds Computer to Crack DES	
Smart Card Tamper Resistance--a Cautionary Note	
Cracking Smart Cards with Differential Power Analysis	
Cracking Smart Cards with Fault Analysis.	
Timing Attacks on Implementations of Diffie Hellman, RSA, DSS, and Other Systems	
The Uncensored Walsh Report on Encryption Technologies	
Intelink	The network linking US spook agencies.

Permissive Action Links	How cryptography is used in nuclear weapons.
Handbook of Applied Cryptography	All 15 chapters and index available on-line (see the middle of the page).
Applied Cryptography	Bruce Schneier's book is available on-line also.

- **MIDDLE EAST CONFLICT**

Scenario for War in the Middle East	The House secret report forcasting war between Syria and Israel and the possible use of nuclear weapons.
Leaked Report Says War May Be Near in Mideast	*Platt's Oilgram News* discovers my web page.
The Nukes of Hazard	South Africa, Israel, Iran, Iraq.
The Third Temple: Blueprint for War?	A "simulation" altar.
NSA, Crypto AG, and the Iraq-Iran War	How the NSA helped out Saddam Hussein.
Mossad Agent Faked Syrian "War" Reports	Deception served political agenda?

- **ECONOMIC COMMENTARY**

Here Come the Zombies	How governments turn bankers into rich gamblers.
Credit Lyonnais & L.F. Rothschild Ready to Topple	Bad loans and bodies in the French laundry.
History of the Euro	The single European currency.
The Asian Financial Crisis in Perspective	The bubble rotation.
The Year 2000 Problem in Perspective	Y2K and the end of time as we know it.
The Gold Market, Part 1	Gold in the international financial system. The London Gold Fixing.
The Gold Market, Part 2	Spot gold trading. The London Bullion Market Association.
The Gold Market, Part 3	Gold lease and libor rates. The forward price of gold. Gold swaps.
The Gold Market, Part 4	The World Trade Center bombing and the gold market. Gold futures. Exchange for physicals.
The Gold Market, Part 5	Gold FRAs. Gold interest-rate swaps. Gold interest-rate guarantees.
The Gold Market, Part 6	Gold options. Options as insurance contracts.
And Now, the Financial Apocalypse	When pundits say "the expansion will last forever," the end is near.
Stocks and Bonds Go Down Together	The peculiar economics of Mafia Bonds.
The Collapse of the New World Order	Visions of paradise and the collapse of the Global Free Lunch.
The Credit Crunch	The core of the July-October 1997 crash.
The Eye of the Storm	The Global Free Lunch is back. Oh, Joy!
Copper-Bottom Pricing	How to price commodity spread options with negative strikes.
Chaos and Fractals in Financial Markets, Part 1	
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Chaos and Fractals in Financial Markets, Part 6

- **INFORMATION WARFARE**

The Jim Bell Files	Assassination politics. The wrath of the IRS. Jim Bell's Theorem. And more.
Report of the Defense Science Board Task Force on Information Warfare	
Industrial Espionage Today & Information Wars of Tomorrow (pdf)	
Industrial Espionage Through Social Engineering (pdf)	
Information Warfare, INFOSEC, and Dynamic Information Defense (pdf)	
FSTAC: Financial Services Risk Assessment (pdf)	
FSTAC: Electric Power Risk Assessment	
GAO: Computer Attacks at Department of Defense (pdf)	
Cyberterror and Cyberhype	There may be risk from cyberterror. But are officials this clueless qualified to judge?

- **THE WHITE HOUSE BIG BROTHER DATA BASE**

The White House "Big Brother" Data Base & How Jackson Stephens Precipitated a Banking Crisis	The role of Jackson Stephen's software firm Systematics in creating security holes both in White House and commercial banking software.
The White House Downloaded	And mad as hell about it, too.
Will Filegate Bring Down the FBI?	The FBI turned over about 2045 files to the White House.
Is the WHODB Clinton's Frankenstein Monster?	Bill's computers watch Bill.
Webb Hubbell and Big Brother (WHODB)	Artificial intelligence, a payoff, a PROMIS conversion, a fired Secret Service agent.
Fund-Raiser's Use of White House Database Reported	Secret project keeps track of political donors.
Safire on Clinton's Data Base	Some "mean hacker" set to attack WHODB?
Insight Magazine reports on WHODB	Click on "Investigative Reports", then click on "White House Scandals".

- **MURDER AT WACO**

Waco Holocaust Electronic Museum	First, gas and burn the children. Then bury the evidence with a bulldozer.
Waco: The Illegal Search Warrant	"Zee Big One"--funding time at the BATF.
The Commander at Ft. Hood, Texas, During the Waco Massacre	The mad bomber of Yugoslavia himself.
ATF Agents Go Shooting with David Koresh	Nine days before the first bloody raid it was all buddy-buddy.
Texas Ranger Report on the Branch Davidians	Waco was a professional hit.

- **ARTICLES ABOUT TWA 800**

The Downing of TWA Flight 800	**TWA Flight 800 was taken out at around 7000 feet altitude by an apparent Stinger missile. A terrorist group connected to Syria gave advance warning and said there would be five more.**
Syria and TWA Flight 800	**101 fast reasons to keep on eye on Syria.**
Bill Clinton's Choo-Choo	**The President takes a train ride. The modified warhead on the missile that took down TWA Flight 800.**
The Phosphorus-Headed Missile and TWA Flight 800	**Some details on the missile that took out TWA Flight 800.**
Witnesses Confirm Missile Took Down TWA 800	**The *New York Post* almost gets the story right, but not quite.**
DIA on TWA 800 Missile, by John McCaslin	**Agent of Defense Intelligence Agency speaks to congress.**
TWA 800 Investigation Cover-Up: The Proof	**Ian Goddard shows clear evidence of an FBI cover-up.**
The Syrian Connection to the Dhahran Bombing	**The FBI, as usual, covers up the evidence.**
What May Have Happened to TWA800	**A boat offshore . . .**
TWA 800: "I Saw an Ordnance Explosion"	
Former Chairman of Joint Chiefs of Staff Revives Missile Theory	
Crash Test Dummies	**More on the botched investigation.**
Interim Report on the Crash of TWA Flight 800 and the Actions of the NTSB and the FBI	**The report of Commander William S. Donaldson, USN Ret. in cooperation with Associated Retired Aviation Professionals.**
Clinton Involved in Coverup of TWA800 Terrorist Missile	**Cmdr. William S. Donaldson, III, USN, Ret.**
TWA 800 Shootdown	**The New American interviews William S. Donaldson.**
TWA 800	**Boeing says missile may have downed plane.**

- **ARTICLES RELATED TO OKLAHOMA CITY BOMBING**

Misconduct Allegations Arise in FBI Lab Probe	**FBI falsifies evidence and commits perjury.**
FBI Suspends Oklahoma City Whistle-blower	**You weren't supposed to know about the contaminated evidence.**
Case Against Oklahoma Bomb Suspect Collapses	**Another government sting gone awry?**
Secret Pentagon Report on Oklahoma City Bombing--Evidence of a Inside Job?	**More than a sting gone bad.**
The John Doe Times	**Info on the Oklahoma City bombing.**
Oklahoma City, Government-Paid Neo-Nazis, and the FCC	**One more attempt to bury the truth about the Oklahoma City bombing.**
Rep. Charles Key on the Facts of the Oklahoma Bombing	
DOJ Report on FBI Crime Lab	**How to lie with forensics.**
Writ of Mandamus	**Stephen Jones, attorney for Timothy McVeigh, on the evidence in the OKC bombing.**

Kirk Lyons, Attorney for Andreas Strassmeir, Responds	
John Doe Times Special Edition	Report that Elohim City leader Millar is FBI informant.
ATF Memo says Carol Howe "key" to identifying Elohim City/OKC bomb link	
General Says Four Charges Destroyed Murrah Federal Building	
Niki Deutchman Interview	
Press Conference of Charles Key re OKC Bombing Foreknowledge and the Response	
New Evidence of OKC Bomb Cover-up	

- **THE CRIMES OF THE FBI**

Was Freeh Fired?--Or Is It Only a Glitch in the FBI's Files?	FBI Director Louis Freeh fails to receive a paycheck because the computer says he was terminated, and the 101 Airborne arrives in Saudia Arabia. (On their way to Syria?)
The FBI and Terrorism	More crimes of the FBI.
The Dickheads are Still Desperate	Why Congress should withdraw all funding from the FBI.
DOJ Report on FBI Crime Lab	How to lie with forensics.
Open Letter to Louis Freeh on Montana Drugs	G-men who smuggle drugs.
Interim Report on the Crash of TWA Flight 800 and the Actions of the NTSB and the FBI	The report of Commander William S. Donaldson, USN Ret. in cooperation with Associated Retired Aviation Professionals.
The FBI's Project Megiddo (.pdf)	The FBI's plan for the Apocalypse.
FBI Shutter Speed	FBI tries to censor Y2K movie.
Subversive Instinct	Get your Freeh shit here. The Times Square video is back on the web.

- **ARTICLES ABOUT IRS EMPLOYEES SELLING COCAINE & TAXPAYER FILES**

Is the IRS Dealing Crack?	Get your tax refund in a baggie, and visit two governors going to jail.
IRS Files for Sale	Buy someone's IRS file for $500.
Join the IRS: Deal Crack with Pay!	Cocaine-dealing IRS employees still on the payroll.

- **ARTICLES ABOUT THE CRIMINAL PRESIDENCY**

Bill Clinton's Cocaine Habit	Our President does five lines a day.
Clinton & Iran	A high-level Democratic delegation asks Clinton not to run again for President. Israeli radio says U.S. to bomb Iranian targets.
The Presidential Crisis	The present danger in Presidential politics.
Bill Clinton and the Missing $100 Million	General Noriega, Barry Seal, and Bill Clinton become suspects regarding millions in missing cocaine money.
October Surprises	Clinton, Dole, Starr, Tucker, Patton, Credit Lyonnais, and Pulsar Data Systems may all have October Surprises.
Happy Birthday, Mr President!	Clinton turns fifty.
Bill Clinton's Choo-Choo	The President takes a train ride. The modified warhead on the missile that took down TWA Flight 800.
The Monster Threatens Starr	Temper tantrums.

How to Create a War	First find some Belgian--er, Kurdish babies, then call in Hill & Knowlton.
Clinton Flips a Coin and Bombs Iraq	Turning the military budget into a political slush fund.
Brother Bill's Bad Week	Clinton rearranges Iraqi sand, the FBI sucks wind, and the crematorium at Mena lurks in the shadows.
Hillary Clinton Puts Out a Contract on Dick Morris	Hillary plots to murder her competition, using government thugs from I-3.
Top Ten Reasons Bill Clinton Resigned	Read these, then make up your own.
Clinton in a Box	Clinton has lost the war, both home and abroad.
What Happened to the Football?	Did someone take the nuclear trigger away from Bill Clinton?
Tenacious Tentacles	The cozy world of Bill Clinton, Jackson Stephens, and Mochtar Riady.
Hillary Clinton, Lafarge, and Kennametal	Hillary helps Richard Mellon Scaife and Jackson Stephens arm Iraq.
Hail to the Cokehead	Walking a mile in Bill Clinton's shoes.
What Election?	Nothing has changed with respect to Clinton's problems.
What if Clinton Never Returned?	Some wishful thinking.
More Top Ten Reasons Bill Clinton Resigned	Fun things to say at the Inaugural Ball.
Sex with Hillary	What one has to do to get a press conference.
Kenneth Starr's Departure Could be Clintons' Worst Nightmare	By John Crudele.
The White House Cult	Now it can be told.
Mexican Madness	Pendejo in paradise.
London Sunday Times: Bonking Bill	
Campaign Finance Report	
The Starr Report	
Hyde's 81 Questions	
Starr's Testimony to the House Judiciary Committee	
Follow the Money	
The Senate Trial	
Clinton Body Count Links	

- **RON BROWN**

Ron Brown's Loose Lips Seal His Fate	More than meets the eye in the crash of Brown's plane.
Database of Ron Brown's Foreign Trade Missions	
The Botched Ron Brown Investigation:	An Interview with AFIP Forensic Pathologist Kathleen Janoski, by Wesley Phelan.

- **MISCELLANEOUS**

Saint Colby and the Fifth Column	Damage control on the Vince Foster story, and why the death of former CIA director William Colby is not an occasion for tears.
Some Observations on the Non-News	The dereliction of the journalism profession, the FBI can't count files, the Montana drug operation, NSA attacks on my Internet posts, and the heroic efforts of Kenneth Starr.

An Apology and Goodbye	A satire directed at a group of super-critics who have spent time attacking or distorting the Vince Foster story as told by Jim Norman and me. (I simply collected some of the arguments and put them in a single post, so everyone could see how silly they sounded.) However, some brain-dead individuals apparently believed my tongue-in-cheek statement that Mike McCurry of the White House Press Office calls me once a week to direct my efforts, and wrote a lot of angry letters condemning everything but their own inattention and gullibility.
How to Launder Money in the Copper Market (1)	Sumitomo services the heroin trade. Intended to become a series.
Woodward's Wayward Book	Why Bob Woodward's book The Choice is particularly ill-timed.
The Clinton Crash	The Clinton administration has imprudently taken credit for much that is taking place in the economy. How would they deal with a stock market crash? Also mentions the banking problems created by alterations in the money flows associated with the drug and arms trade.
Is the CIA Trying for a Piece of Fifth Column Action?	*Contact* fails to contact the Fifth Column.
The Governor of Kentucky to be Indicted Soon?	Operation BOPTROT and Casey's plan to turn Russian soldiers into drug addicts.
Dole Dumps an Old Friend and Lies About His Finances	Why Dole resembles Clinton.
The Cracks in Clinton's Economy	John Deutch and Alan Greenspan worry about stock declines.
The Fifth Column Gets Press	And the Prez worries Dick Morris might squeal.
The Uses of Terrorism	The trade-off between guns and buttering-up-voters.
The Sniffles of Susan	Susan McDougal, following bad legal advice, becomes a martyr without a cause.
The Symbiosis of Alan Greenspan and Bill Clinton	Why is the Federal Reserve illegally intervening in the stock market?
Pinnacle Bank--The Usual Suspects	Banking, the Arkansas way.
Wars and Rumors of War	Iraq, I3, and why Bill won't pardon Susan.
Musings	Presidential pardons and all that.
Bill's Blow, Stock Blowoffs, and Millennial Madness	The future sucks.
Five Indictments of the Mass Media	The alcoholic press consumes a fifth.
The Dickheads Are Getting Desperate	Federal agents show up in Nancy, Kentucky.
Laundering Numbers	Sumitomo & FBI files: the media can't count.
The Starr Detractors	As the worm squirms. Another chorus of "no more indictments".
The Other Starr Detractors	Chris Ruddy and Ambrose Evans-Pritchard.
The Joke Is on You	The media releases us from the burden of political truth.
October Country	That country composed in the main of cellars, sub-cellars, coal-bins, closets, attics.
General Convicted for Political Fund-Raising	The politics of promotion.

The Coke Was Stored in Hangars 4 and 5	Cocaine distribution from the Ilo Pango Air Force Base, and the roles of the CIA, NSA, and the Mossad.
The Starr Indictments: The Media Wrong Again	And that's not all the media is wrong about.
Bye Bye, Miss American Pie	Lemmings and stock market calculations.
Al Gore as President	And Tipper as FLOTUS.
Sticky Fingers at the Justice Department and U.S. Customs	The government steals and doesn't pay its bills.
Earl Brian Convicted in California	The chief marketeer of the PROMIS software is convicted of financial fraud.
The Starr Indictments II	Trick or treat.
Murder and Drugs in Arkansas	The FBI sits on evidence in a drug-related murder.
Ten Predictions & Postmortem	My March 1996 predictions and postmortem comment.
Twenty Predictions for 1997	Sixscore thousand persons that cannot discern between their right hand and their left hand; and also much cattle.

Charles "the Angel of Death" Hayes

The "Angel of Death" is an Internet term applied to Charles S. Hayes because of his role in bringing about the resignation of many a corrupt politician. Hayes is also the head and public representative of the Fifth Column group of hackers who track political payoffs through money-laundering channels.

Angel of Death Gives Deposition to DOJ in Inslaw Case	An argument for the resignation of the entire Justice Department.
The Dickheads Are Getting Desperate	Federal agents show up in Nancy, Kentucky.
Sticky Fingers at the DOJ and U.S. Customs	The government steals and doesn't pay its bills.
Earl Brian Convicted in California	The chief marketeer of the PROMIS software is convicted of financial fraud.
Angel of Death Arrested	Don't talk about PROMIS. Don't help the Special Prosecutor.
Chuck Hayes Versus Bill Clinton	The progress of the war of Clinton's resignation.
Chuck Hayes Versus Bill Clinton 2	FBI director Louis Freeh doesn't like Chuck Hayes.
Chuck Hayes and Me, Part 1	The House Committee on Banking asks the NSA about PROMIS software and money-laundering.
Chuck Hayes and Me, Part 2	Hayes investigates drug-money laundering, while the House Committee on Banking kisses NSA butt.
Chuck Hayes and Me, Part 3	Why the Fifth Column is necessary: "I don't want another Danny Casolaro on my hands.
The Metaphysics of Political Illusion	U.S. Attorney Joseph Famularo is a kook.
Orlin Grabbe Versus Joseph Famularo	My letter to U.S. Attorney Joseph Famularo, demanding reasons for his falsified claims.
Orlin Grabbe Versus Joseph Famularo 2	Famularo's chief witness is also a liar.
Orlin Grabbe Versus Joseph Famularo 3	Famularo says it's all Martin Hatfield's fault.
The FBI Conspiracy Against Chuck Hayes	FBI agent David Keller overrules the Court and the Assistant US Attorney.

Letter to Chuck Hayes	
Court Testimony Regarding Lawrence Myers	Why is this convicted thief and psychiatric case running free while Chuck Hayes is in jail?
Charles S. Hayes Sentencing Transcript	Judge Jenny places presentence investigation report under seal!
James Norman's Letter to Warden of Manchester Prison	
New Address for Chuck Hayes	
Judge Jennifer B. Coffman Coaches the Prosecution	How to deal with the testimony of (Convicted Felon) Lawrence Myers and (FBI Agent) Steve Brannon.

On July 25, 1997, despite a presentence investigation report that recommended that Charles Hayes be sentenced to time served (i.e. immediately released), Judge Jennifer B. Coffman sentenced Charles Hayes to the federal maximum of 10 years in prison.

How You Can Help Charles Hayes

Allegations Regarding Vince Foster, the NSA, and Banking Transactions Spying

A series exploring the murder of Vince Foster, who--among other duties-- was an overseer of an NSA project to spy on banking transactions. The series begins with a memo by Jim Norman, Senior Editor at Forbes, to Mike McCurry of the White House Press Office inquiring about the espionage activities of Vince Foster. Then (reflecting my own on-going investigation) it gradually branches out into relevant background issues necessary to understand pieces of the story. Finally, it converges back to Foster's final days.

Most of the people who have an opinion on this issue have never done any investigation themselves, and the few who have long ago buried their heads in the flora of Ft. Marcy Park and never looked at the larger picture.

Foster was under counter-intelligence investigation when he died. Nearly every investigator misses this part of the picture. Did these activities lead to his death? Or did others take advantage of the circumstances to rid themselves of someone who might talk? Neither question can be answered until the actual killer(s) is (are) identified.

Part 1	Jim Norman sends a memo to the White House, which leaks it to a Starr assistant.
Part 2	Charles O. Morgan threatens to sue anyone who says Systematics has a relationship with the NSA.
Part 3	Vince Foster oversaw covert money laundering at Systematics.
Part 4	Hillary Clinton and Web Hubbell represented Systematics during the BCCI takeover of First American.
Part 5	Philosophical musings on the meaning of this investigation.
Part 6	The curious tenacles (and strange denials) of Systematics.
Part 7	Virtual realities in the media and banking. Israel as a virtual nuclear power.
Part 8	Caspar Weinberger's Swiss account. Israel's nuclear spying.
Part 9	Cover-up by House Banking Committee investigators.
Part 10	BCCI, money laundering, and the nuclear weapons programs of Pakistan and Israel.
Part 11	Money laundering and the intelligence community.

Other Foster Resources

America's Dreyfus Affair: The Case of the Death of Vince Foster	Parts 1-6. Essay by David Martin.
Vince Foster and the NSA	Charles R. Smith on Vince Foster and the Clipper Chip.

Parallels Between the Deaths of Tommy Burkett and Vince Foster	Hugh Turley shows there is consistency in the technology of assisted "suicide".
Ft. Marcy Park Witness Patrick Knowlton Lawsuit	Second Amended Complaint, October 1998
Clinton Meets Columbo	Oval office showdown.
The Starr Report with the Knowlton Appendix	Microsoft Word document. Includes exhibits.
Vince Foster FBI Files Online	

Miscellaneous Articles and Essays

My Resume	I do consulting. Unlike the free information on this web page, it comes at a high price.
In Praise of Chaos	Speech given to the Eris Society in August 1993. Since reprinted in *Liberty* (March 1994) and the SIRS Renaissance electronic data base (1995).
Eloge du Chaos	French translation of "In Praise of Chaos"
Cuba de mi Amor	Music to your ears.
Memories of Pasadena	A personal memoir. Appeared in *Laissez Faire City Times*, Vol 5, No 2, Jan. 8, 2001.

Short Stories

Keys	Published in *Laissez Faire City Times*, Vol 2, No 34, Oct 19, 1998.
Agrarian Life	Published in *Laissez Faire City Times*, Vol 2, No 35, Oct 26, 1998.
A Hundred Eighty Dollars	Published in *Laissez Faire City Times*, Vol 2, No 36, Nov. 2, 1998.
Karma Accountant	Problems in spiritual reckoning. Published in *Liberty* (August 1993).
Cover Girl	Published in *Laissez Faire City Times* (November 1997).
The Hat	How to Survive in the East Village. Appeared in *Art Times* (November 1992).
Connections	The relationship between sex and quantum mechanics. Generated some fan mail from physicists. First appeared in *Beet #6* (Spring 1992).
Feed the Children	Saving the world and saving oneself. Published in *Liberty* (May 1995).
Dolphin Man	Published in *Laissez Faire City Times*, Vol 3, no 40, Oct. 11, 1999.
The Age of the Feuilleton	Published in *Laissez Faire City Times*, Vol 3, no 42, Oct. 25, 1999.
Waiting for Gödel	Published in *Laissez Faire City Times*, Vol 3, no 46, Nov. 29, 1999.

That's All Folks!

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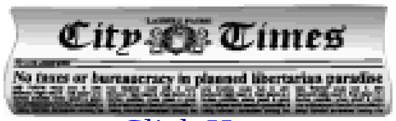

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Chaos and Fractals in Financial Markets

Part 2

by J. Orlin Grabbe

The French Gambler and the Pollen Grains

In 1827 an English botanist, Robert Brown got his hands on some new technology: a microscope "made for me by Mr. Dolland, . . . of which the three lenses that I have generally used, are of a 40^{th}, 60^{th}, and 70^{th} of an inch focus."

Right away, Brown noticed how pollen grains suspended in water jiggled around in a furious, but random, fashion.

To see what Brown saw under his microscope, make sure that Java is enabled on your web browser, and then click here.

What was going on was a puzzle. Many people wondered: Were these tiny bits of organic matter somehow ***alive***? Luckily, Hollywood wasn't around at the time, or John Carpenter might have made his wonderful horror film *They Live!* about pollen grains rather than about the infiltration of society by liberal control-freaks.

Robert Brown himself said he didn't think the movement had anything to do with tiny currents in the water, nor was it produced by evaporation. He explained his observations in the following terms:

> "That extremely minute particles of solid matter, whether obtained from organic or inorganic substances, when suspended in pure water, or in some other aqueous fluids, exhibit motions for which I am unable to account, and from which their irregularity and seeming independence resemble in a remarkable degree the less rapid motions of some of the simplest animalcules of infusions. That the smallest moving particles observed, and which I have termed Active Molecules, appear to be spherical, or nearly so, and to be between 1-20,0000dth and 1-30,000dth of an inch in diameter; and

> that other particles of considerably greater and various size, and either of similar or of very different figure, also present analogous motions in like circumstances.
>
> "I have formerly stated my belief that these motions of the particles neither arose from currents in the fluid containing them, nor depended on that intestine motion which may be supposed to accompany its evaporation."[1]

Brown noted that others before him had made similar observations in special cases. For example, a Dr. James Drummond had observed this fishy, erratic motion in fish eyes:

> "In 1814 Dr. James Drummond, of Belfast, published in the 7th Volume of the Transactions of the Royal Society of Edinburgh, a valuable Paper, entitled 'On certain Appearances observed in the Dissection of the Eyes of Fishes.'
>
> "In this Essay, which I regret I was entirely unacquainted with when I printed the account of my Observations, the author gives an account of the very remarkable motions of the spicula which form the silvery part of the choroid coat of the eyes of fishes."

Today, we know that this motion, called ***Brownian motion*** in honor of Robert Brown, was due to random fluctuations in the number of water molecules bombarding the pollen grains from different directions.

Experiments showed that particles moved further in a given time interval if you raised the temperature, or reduced the size of a particle, or reduced the "viscosity" [2] of the fluid. In 1905, in a celebrated treatise entitled *The Theory of the Brownian Movement* [3], Albert Einstein developed a mathematical description which explained Brownian motion in terms of particle size, fluid viscosity, and temperature. Later, in 1923, Norbert Wiener gave a mathematically rigorous description of what is now referred to as a "stochastic process." Since that time, Brownian motion has been called a *Wiener process*, as well as a "diffusion process", a "random walk", and so on.

But Einstein wasn't the first to give a mathematical description of Brownian motion. That honor belonged to a French graduate student who loved to gamble. His name was Louis Bachelier. Like many people, he sought to combine duty with pleasure, and in 1900 in Paris presented his doctoral thesis, entitled *Théorie de la spéculation*.

What interested Bachelier were not pollen grains and fish eyes. Instead, he wanted to know why the *prices* of stocks and bonds jiggled around on the Paris bourse. He was particularly intrigued by bonds known as *rentes sur l'état*— perpetual bonds issued by the French government. What were the laws of this jiggle? Bachelier wondered. He thought the answer lay in the prices being bombarded by small bits of news. ("The British are coming, hammer the prices down!")

The Square Root of Time

Among other things, Bachelier observed that the ***probability intervals*** into which prices fall seemed to increased or decreased with the square-root of time ($T^{0.5}$). This was a key insight.

By "probability interval" we mean a given probability for a range of prices. For example, prices might fall within a certain price range with 65 percent probability over a time period of one year. But over two years, the same price range that will occur with 65 percent probability will be larger than for one year. How much larger? Bachelier said the change in the price range was proportional to the square root of time.

Let P be the current price. After a time T, the prices will (with a given probability) fall in the range

($P - \mathbf{a}\ T^{0.5}$, $P + \mathbf{a}\ T^{0.5}$), for some constant **a**.

For example, if T represents one year (T=1), then the last equation simplifies to

($P - \mathbf{a}$, $P + \mathbf{a}$), for some constant **a**.

The price variation over two years (T=2) would be

$\mathbf{a}\ T^{0.5} = \mathbf{a}(2)^{0.5} = 1.4142\ \mathbf{a}$

or 1.4142 times the variation over one year. By contrast, the variation over a half-year (T=0.5) would be

$\mathbf{a}\ T^{0.5} = \mathbf{a}(0.5)^{0.5} = .7071\ \mathbf{a}$

or about 71 percent of the variation over a full year. That is, after 0.5 years, the price (with a given probability) would be in the range

(P –.7071**a** , P + .7071**a**).

Here the constant **a** has to be determined, but one supposes it will be different for different types of prices: **a** may be bigger for silver prices than for gold prices, for example. It may be bigger for a share of Yahoo stock than for a share of IBM.

The range of prices for a given probability, then, depends on the constant a, and on the square root of time ($T^{0.5}$). This was Bachelier's insight.

Normal Versus Lognormal

Now, to be sure, Bachelier made a financial mistake. Remember (from Part 1 of this series) that in finance we always take logarithms of prices. This is for many reasons. Most changes in most economic variables are *proportional* to their current level. For example, it is plausible to think that the variation in gold prices is proportional to the level of gold prices: $800 dollar gold varies in greater increments than does gold at $260.

The change in price, ΔP, as a proportion of the current price P, can be written as:

$\Delta P/P$.

But this is approximately the same as the change in the log of the price:

$\Delta P/P \approx \Delta (\log P)$.

What this means is that Bachelier should have written his equation:

($\log P - \mathbf{a}\, T^{0.5}$, $\log P + \mathbf{a}\, T^{0.5}$), for some constant **a**.

However, keep in mind that Bachelier was making innovations in both finance and in the mathematical theory of Brownian motion, so he had a hard enough time getting across the basic idea, without worrying about fleshing out all the correct details for a non-existent reading audience. And, to be sure, almost no one read Bachelier's PhD thesis, except the celebrated mathematician Henri Poincaré, one of his instructors.

The range of prices for a given probability, then, depends on the constant **a**, and on the square root of time ($T^{0.5}$), ***as well as the current price level P***.

To see why this is true, note that the probability range for the *log* of the price

(log P –**a** $T^{0.5}$, log P + **a** $T^{0.5}$)

translates into a probability range for the *price itself* as

(P exp(- **a** $T^{0.5}$), P exp(**a** $T^{0.5}$)) .

(Here "exp" means exponential, remember? For example, exp(-.7) = $e^{-.7}$ = $2.718281^{-.7}$ = .4966.)

Rather than adding a plus or minus something to the current price P, we multiply something by the current price P. So the answer depends on the level of P. For a half-year (T=0.5), instead of

(P –.7071**a** , P + .7071**a**)

we get

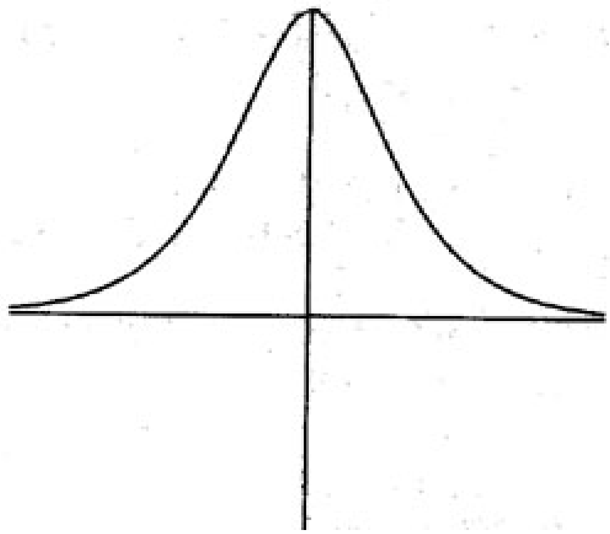
(P exp(- .7071 **a**), P exp(.7071 **a**)) .

The first interval has a constant width of 1.4142 **a**, no matter what the level of P (because P + .7071 **a** - (P -.7071 **a**) = 1.4142 **a**). But the width of the second interval varies as P varies. If we double the price P, the width of the interval doubles also.

Bachelier allowed the price range to depend on the constant **a** and on the square root of time ($T^{0.5}$), but omitted the requirement that the range should also depend on the current price level P.

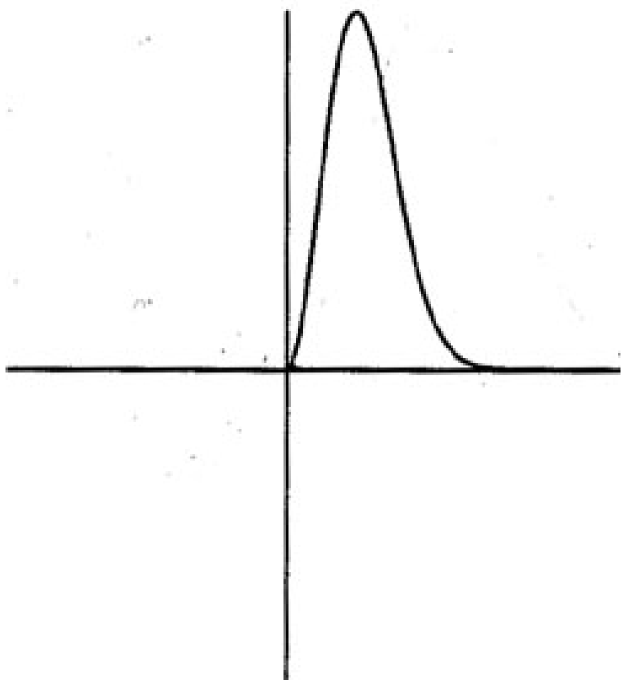
The difference in the two approaches is that if price increments (ΔP) are independent, and have a finite variance, then the price P has a ***normal*** (Gaussian distribution). But if increments in the log of the price (Δ log P) are independent, and have a finite variance, then the price P has a ***lognormal*** distribution.

Here is a picture of a normal or Gaussian distribution:



The left-hand tail never becomes zero. No matter where we center the distribution (place the mean), there is always positive probability of negative numbers.

Here is a picture of a lognormal distribution:



The left-hand tail of a lognormal distribution becomes zero at zero. No matter where we center the distribution (place the mean), there is zero probability of negative numbers.

A lognormal distribution assigns zero probability to negative prices. This makes us happy because most businesses don't charge negative prices. (However, US Treasury bills paid negative interest rates on certain occasions in the 1930s.) But a normal distribution assigns positive probability to negative prices. We don't want that.

So, at this point, we have seen Bachelier's key insight that probability intervals for prices change proportional to the square root of time (that is, the probability interval around the current price P changes by **a** $T^{0.5}$), and have modified it slightly to say that probability intervals for the *log of prices* change proportional to the square root of time (that is, the probability interval around log P changes by **a** $T^{0.5}$).

How Big Is It?

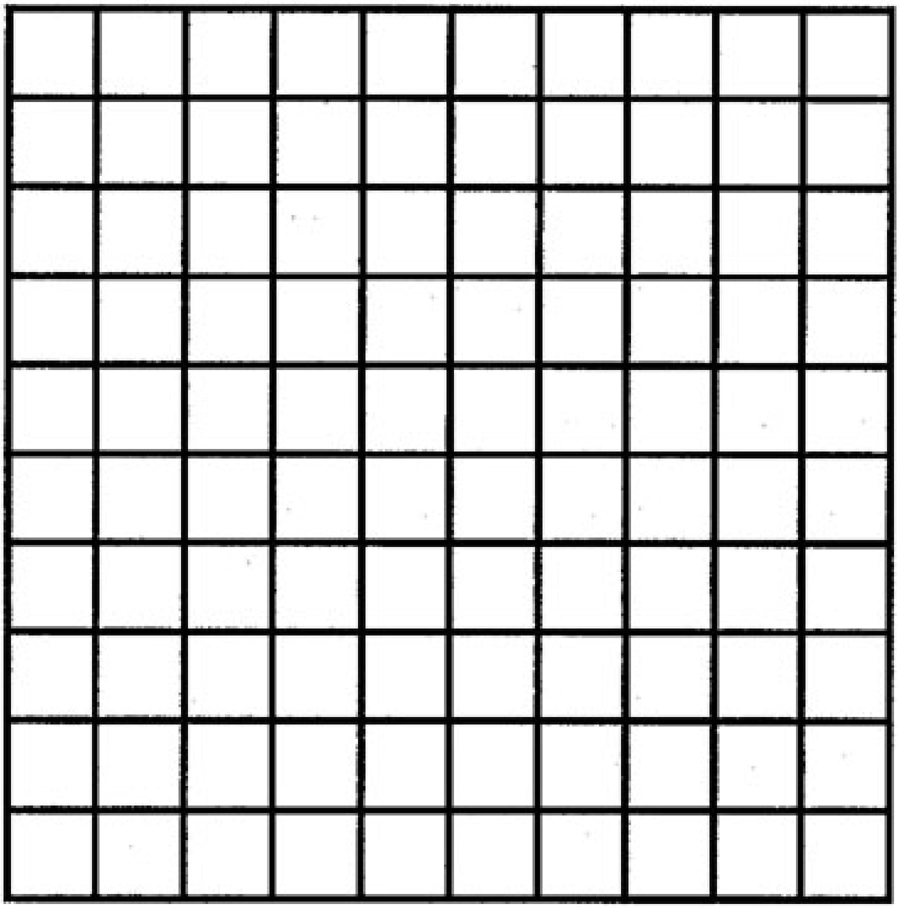
Now we are going to take a break from price distributions, and pursue the question of how we measure things. How we measure length, area, volume, or time. (This will lead us from Bachelier to Mandelbrot.)

Usually, when we measure things, we use everyday dimensions (or at least the ones we are familiar with from elementary plain geometry). A point has zero dimension. A line has one dimension. A plane or a square has two dimensions. A cube has three dimensions. These basic, common-sense type dimensions are sometimes referred to as ***topological dimensions***.

We say a room is so-many "square feet" in size. In this case, we are using the two-dimensional concept of area. We say land is so-many "acres" in size. Here, again, we are using a two-dimensional concept of area, but with different units (an "acre" being 43,560 "square feet"). We say a tank holds so-many "gallons". Here we are using a measure of volume (a "gallon" being 231 "cubic inches" in the U.S., or .1337 "cubic feet").

Suppose you have a room that is 10 feet by 10 feet, or 100 square feet. How much carpet does it take to cover the room? Well, you say, a 100 square feet of carpet, of course. And that is true, for ordinary carpet.

Let’s take a square and divide it into smaller pieces.
Let’s divide each side by 10:



We get 100 pieces. That is, if we divide by a scale factor of 10, we get 100 smaller squares, all of which look like the big square. If we multiply any one of the smaller squares by 10, we get the original big square.

Let’s calculate a dimension for this square. We use the same formula as we used for the Sierpinski carpet:

$N = r^D$.

Taking logs, we have log N = D log r, or D = log N/ log r.

We have N = 100 pieces, and r = 10, so we get the dimension D as

D = log(100)/log(10) = 2.

(We are using "log" to mean the natural log, but notice for this calculation, which involves the ratio of two logs, that it doesn’t matter what base we use. You can use logs to the base 10, if you wish, and do the calculation in your head.)

We called the dimension D calculated in this way (namely**, by comparing the number of similar objects N we got at different scales to the scale factor r**) a *Hausdorff dimension*. In this case, the Hausdorff dimension 2 is the same as the ordinary or topological dimension 2.

So, in any case, the dimension is 2, just as you suspected all along. ***But suppose you covered the floor with Sierpinski carpet. How much carpet do you need then?***

We saw (in Part 1) that the Sierpinski carpet had a Hausdorff dimension D = 1.8927… A Sierpinski carpet which is 10 feet on each side would only have $N = 10^{1.8927} =$ 78.12 square feet of material in it.

Why doesn't a Sierpinski carpet with 10 feet on each side take 100 square feet of material? Because the Sierpinski carpet *has holes in it*, of course.

Remember than when we divided the side of a Sierpinski carpet by 3, we got only 8 copies of the original because we threw out the center square. So it had a Hausdorff dimension of D = log 8/ log 3 = 1.8927. Then we divided each of the 8 copies by 3 again , threw out the center squares once more, leaving 64 copies of the original. Dividing by 3 twice is the same as dividing by 9, so, recalculating our dimension, we get D = log 64/ log 9 = 1.8927.

An ordinary carpet has a Hausdorff dimension of 2 and a topological (ordinary) dimension of 2. A Sierpinski carpet has a Hausdorff dimension of 1.8927 and a topological dimension of 2. [4]

Benoit Mandelbrot defined a ***fractal*** as **an object whose Hausdorff dimension is different from its topological dimension**. So a Sierpinski carpet is a fractal. An ordinary carpet isn't.

Fractals are cheap and sexy. A Sierpinski carpet needs only 78.12 square feet of material to cover 100 square feet of floor space. Needing less material, a Sierpinski carpet costs less. Sure it has holes in it. But the holes form a really neat pattern. So a Sierpinski carpet is sexy. Cheap and sexy. You can't beat that.

History's First Fractal

Let's see if we have this fractal stuff straight. Let's look at the first known fractal, created in 1870 by the mathematical troublemaker George Cantor.

Remember that we create a fractal by forming similar patterns at different scales, as we did with the Sierpinski carpet. It's a holey endeavor. In order to get a carpet whose Hausdorff dimension was less than 2, we created a pattern of holes in the carpet. So we ended up with an object whose Hausdorff dimension D (which compares the number N of different, but similar, objects at different scales r, $N = r^D$) was more than 1 but less than 2. That made the Sierpinski carpet a fractal, because its Hausdorff dimension was different from its topological dimension.

What George Cantor created was an object whose dimension was more than 0 but less than 1. That is, a holey object that was more than a point (with 0 dimensions) but less than a line (with 1 dimension). It's called Cantor dust. When the Cantor wind blows, the dust gets in your lungs and you can't breathe.

To create Cantor dust, draw a line and cut out the middle third:

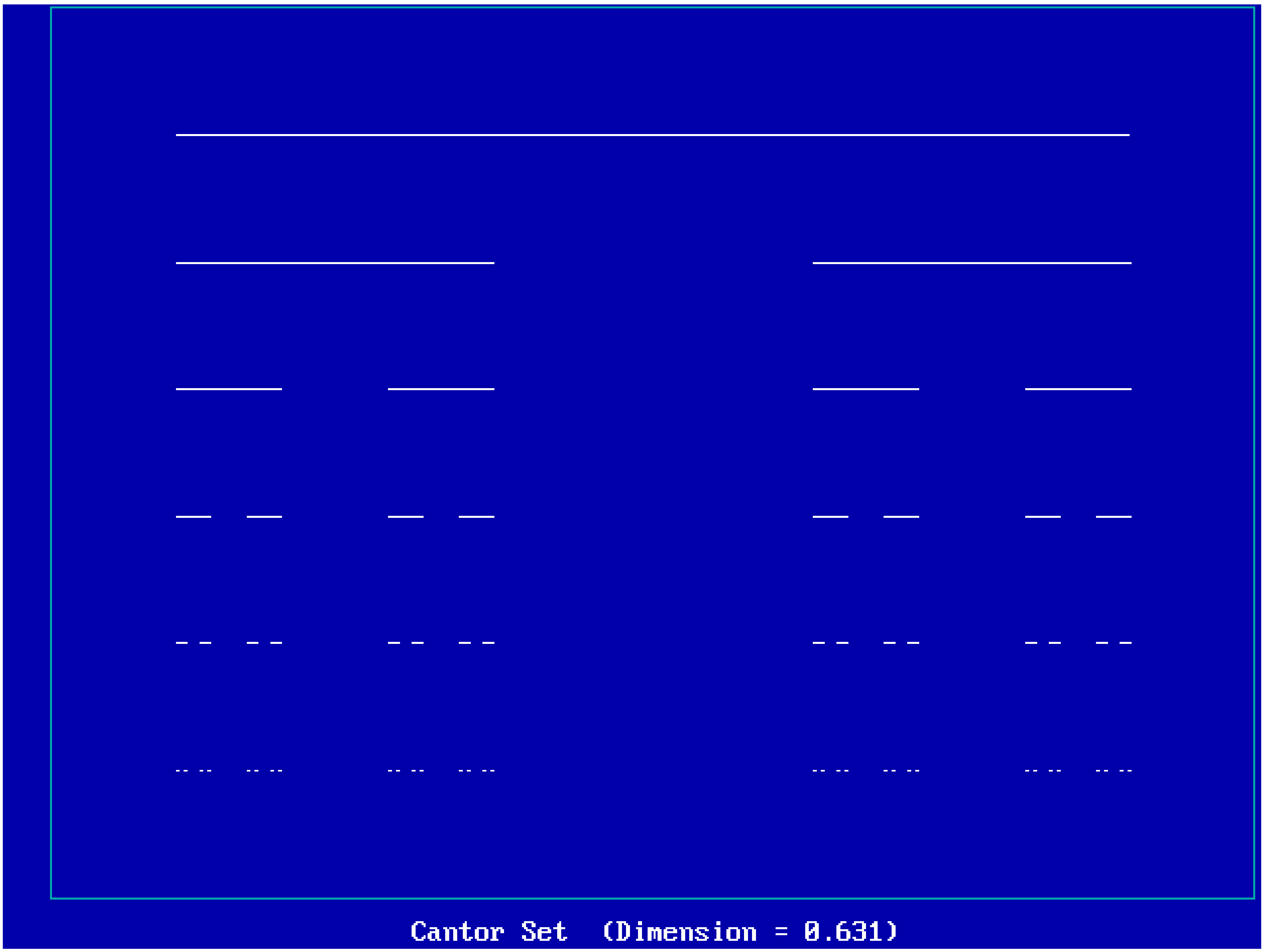
```
0____________________________________________________________1
```

```
0__________________1/3               2/3_________________1
```

Now cut out the middle thirds of each of the two remaining pieces:

```
0____1/9   2/9____ 1/3               2/3____7/9  8/9 ____1
```

Now cut out the middle thirds of each of the remaining four pieces, and proceed in this manner for an infinite number of steps, as indicated in the following graphic.



Cantor Set (Dimension = 0.631)

What's left over after all the cutting is *Cantor dust*.

At each step we changed *scale* by r = 3, because we divided each remaining part into 3 pieces. (Each of these pieces had 1/3 the length of the original part.) Then we threw away the middle piece. (That’s how we created the holes.) That left 2 pieces. At the next step there were 4 pieces, then 8, and so on. At each step *the number of pieces* increased by a factor of N = 2. Thus the Hausdorff dimension for Cantor dust is:

D = log 2 / log 3 = .6309.

Is Cantor dust a fractal? Yes, as long as the topological dimension is different from .6309, which it surely is.

But—what is the topological dimension of Cantor dust? We can answer this by seeing how much of the original line (with length 1) we cut out in the process of making holes.

At the first step we cut out the middle third, or a length of 1/3. The next step we cut out the middle thirds of the two remaining pieces, or a length of 2(1/3)(1/3). And so on. The total length cut out is then:

$1/3 + 2(1/3^2) + 4(1/3^3) + 8(1/3^4) + \ldots = 1.$

We cut out all of the length of the line (even though we left an infinite number of points), so the Cantor dust that's left over has length zero. Its topological dimension is zero. Cantor dust is a fractal with a Hausdorff dimension of .6309 and a topological dimension of 0.

Now, the subhead refers to Cantor dust as "history’s first fractal". That a little anthropocentric. Because nature has been creating fractals for millions of years. In fact, most things in nature are not circles, squares, and lines. Instead they are fractals, and the creation of these fractals are usually determined by chaos equations. Chaos and fractal beauty are built into the nature of reality. Get used to it.

> Today, there are roughly of order 10^3 recognized fractal systems in nature, though a decade ago when Mandelbrot's classic *Fractal Geometry of Nature* was written, many of these systems were not known to be fractal. [5]

Fractal Time

So far we’ve seen that measuring things is a complicated business. Not every length can be measured with a tape measure, nor the square footage of material in every carpet measured by squaring the side of the carpet.

Many things in life are fractal, and follow power laws just like the D of the Hausdorff dimension. For example, the "loudness" L of noise as heard by most humans is proportional to the sound intensity I raised to the fractional power 0.3:

$L = a\ I^{0.3}$.

Doubling the loudness at a rock concert requires increasing the power output by a factor of ten, because

$a\ (10\ I)^{0.3} = 2\ a\ I^{0.3} = 2\ L$.

In financial markets, another subjective domain, "time" is fractal. Time does not always move with the rhythms of a pendulum. Sometimes time is less than that. In fact, we've already encounted fractal time with the Bachelier process, where the log of probability moved according to

$\mathbf{a}\ T^{0.5}$.

Bachelier observed that if the time interval was multiplied by 4, the probability interval only increased by 2. In other words, at a scale of $\mathbf{r} = 4$, the number N of similar probability units was N = 2. So the Hausdorff dimension for time was:

$D = \log N / \log r = \log 2 / \log 4 = 0.5$.

In going from Bachelier to Mandelbrot, then, the innovation is not in the observation that time is fractal: that was Bachelier's contribution. Instead the question is: What is the correct fractal dimension for time in speculative markets? Is the Hausdorff dimension really D = 0.5, or does it take other values? And if the Hausdorff dimension of time takes other values, what's the big deal, anyway?

The way in which Mandlebrot formulated the problem provides a starting point:

> Despite the fundamental importance of Bachelier's process, which has come to be called "Brownian motion," it is now obvious that it does not account for the abundant data accumulated since 1900 by empirical economists, simply because *the empirical distributions of price changes are usually too "peaked" to be relative to*

samples from Gaussian populations. [6]

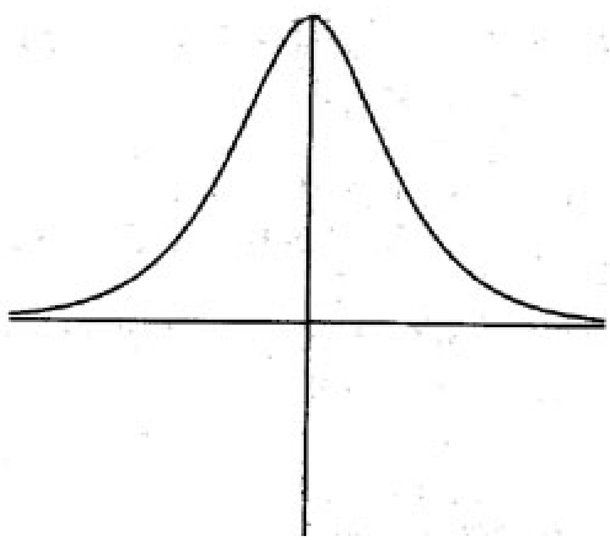
What does Mandelbrot mean by "peaked"? It's now time for a discussion of probability.

Probability is a One-Pound Jar of Jelly

Probability is a one-pound jar of jelly. You take the jelly and smear it all over the real line. The places where you smear more jelly have more probability, while the places where you smear less jelly have less probability. Some spots may get no jelly. They have no probability at all—their probability is zero.

The key is that you *only have one pound* of jelly. So if you smear more jelly (probability) at one location, you have to smear less jelly at another location.

Here is a picture of jelly smeared in the form of a bell-shaped curve:



The jelly is smeared between the horizontal (real) line all the way up to the curve, with a uniform thickness. The result is called a "standard normal distribution". ("Standard" because its mean is 0, and the standard deviation is 1.) In this picture, the point where the vertical line is and surrounding points have the jelly piled high—hence they are more probable.

As we observed previously, for the normal distribution jelly gets smeared on the real (horizontal) line all the way to plus or minus infinity. There may not be much jelly on the distant tails, but there is always some.

Now, let's think about this bell-shaped picture. What does Mandelbrot mean by the distribution of price changes being "too peaked" to come from a normal distribution?

Does Mandelbrot's statement make any sense? If we smear more jelly at the center of the bell curve, to make it taller, we can only do so by taking jelly from some other place. Suppose we take jelly out of the tails and intermediate parts of the distribution and pile it on the center. The distribution is now "more peaked". It is more centered in one place. It has a smaller standard deviation—or smaller dispersion around the mean. But—it could well be still normal.

So what's with Mandelbrot, anyway? What does he mean? We'll discover this in Part 3 of this series.

Click here to see the Answer to Problem 1 from Part 1. The material therein should be helpful in solving **Problem 2**.

Meanwhile, here are two new problems for eager students:

Problem 3: Suppose you create a Cantor dust using a different procedure. Draw a line. Then divide the line into 5 pieces, and throw out the second and fourth pieces. Repeat this procedure for each of the remaining pieces, and so on, for an infinite number of times. What is the fractal dimension of the Cantor dust created this way? What is its topological dimension? Did you create a new fractal?

Problem 4: Suppose we write all the numbers between 0 and 1 in ternary. (Ternary uses powers of 3, and the numbers 0, 1, 2. The ternary number .1202, for example, stands for 1 x 1/3 + 2 x 1/9 + 0 x 1/27 + 2 x 1/81.) Show the Cantor dust we created here in Part 2 (with a Hausdorff dimension of .6309) can be created by taking all numbers between 0 and 1, and eliminating those numbers whose ternary expansion contains a 1. (In other words, what is left over are all those numbers whose ternary expansions only have 0s and 2s.)

And enjoy the fractal:

Notes

[1] Robert Brown, "Additional Remarks on Active Molecules," 1829.

[2] Viscosity is a fluid's stickiness: honey is more viscous than water, for example. "Honey don't jiggle so much."

[3] I am using the English title of the well-known Dover reprint: *Investigations on the Theory of the Brownian Movement*, Edited by R. Furth, translated by A.D. Cowpter, London, 1926. The original article was in German and titled somewhat differently.

[4] I am admittingly laying a subtle trap here, because of the undefined nature of "topological dimension". This is partially clarified in the discussion of Cantor dust, and further discussed in Part 3.

[5] H. Eugene Stanley, *Fractals and Multifractals*, 1991

[6] Benoit Mandelbrot, "The Variation of Certain Speculative Prices," *Journal of Business*, 36(4), 394-419, 1963.

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from *The Laissez Faire City Times*, Vol 3, No 24, June 14, 1999

What Robert Brown Saw Under His Microscope

"true" if statistics of random walk are to be shown size in pixels of the simulation-area

The applet above (written by Anne M. Denton) shows a small circle inside of a large circle. Think of the large circle as Robert Brown's microscope lens. The smaller circle inside is one of the pollen grains being observed by Brown. The red line is the center of the circle. The pollen grain is being bombarded by water molecules, which causes it to move about erratically.

Actually, however, Brown couldn't see the water molecules, and it would not be until 1905 that it was accepted that the erratic movement was caused by molecular bombardment. So to *really* see what Brown saw, click on "flip view" above , and you will just see the red center moving erratically. That's what Brown and others saw. What kind of strange animal was this?

Using the buttons above, you can stop and start the applet, or reset it. You can also change the parameters, such as mu (which is the viscosity of the fluid). If you slide mu to the left, so that the fluid becomes more like water than like honey, the particle moves about more freely.

In 1905 Einstein showed that the distance r traveled by the particle was proportional to the square root of time T:

$r = a\ T^{0.5}$.

So on a log r versus log T scale, we get a line with a slope of 0.5.

Return to Part 2

Chaos and Fractals in Financial Markets

by J. Orlin Grabbe

Answer to Exercise 1

Exercise 1: Iterate the following system: x(n+1) = 2x(n) mod 1. [By "mod 1" is meant that only the fractional part of the result is kept. For example, 3.1416 mod 1 = .1416.] Is this system chaotic?

To get a feel for how the system behaves, let's first iterate a few values. Start with **x(0) = .1**. We get x(1) = 2(.1) mod 1 = .2 mod 1 = .2 . This is the first iteration. This and the following iterations are listed in the table:

Iteration	Value of x
1	.2
2	.4
3	.8
4	.6
5	.2
6	.4
7	.8
8	.6
9	.2
10	.4
11	.8

And so on. The values of **x** cycle through .6, .2, .4, .8, over and over.

Now let **x(0) = 1.25**. What values to we get? The first iteration is x(1) = 2(1.25) mod 1 = 2.50 mod 1 = .50. The second iteration is x(2) = 2(.5) mod 1 = 1 mod 1 = 0. The third iteration is x(3) = 2(0) mod 1 = 0 mod 1 = 0. Once at zero, the system stays there.

Iteration	Value of x
1	.50
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	0
10	0
11	0

Because all values are taken mod 1, we drop (chop off) anything to the left of the decimal point. **So we are really only concerned with numbers between 0 and 1**.

Suppose we write each of the numbers between 0 and 1, not in the decimal system, but in the binary system. The binary system uses only 0s and 1s, which represent powers of 2.

Consider, for example, the binary number

.1101 .

The first place to the right of the "decimal point" represents ½. The next place represents ¼ . The third place represents 1/8 . And so on. The n-th place represents $1/2^n$.

So

.1101 = 1 x ½ + 1 x ¼ + 0 x 1/8 + 1 x 1/16

= ½ + ¼ + 1/16 = 13/16.

The binary

.0011 = 0 x ½ + 0 x ¼ + 1 x 1/8 + 1 x 1/16 = 3/16.

Now, this may look like we've made things more complicated, but actually we've made them extremely simple. What happens when we multiply by 2? Well, since the n-th decimal place represents $1/2^n$, if we multiply it by 2, we have

$2 \times 1/2^n = 1/2^{n-1}$.

This is the (n-1)-th place to the right of the decimal point, or one place closer.

So take .1101. To multiply it by 2, we move the decimal point one place to the right and get

1.101 .

We then apply mod 1, which chops off everything to the left of the decimal point and get

.101 .

Multiply by 2 again, we get

.01.

Then multiply by 2 again, we get

.1

Finally, one more multiplication, and we get

0.

So the first observation we can make about the system x(n+1) = 2 x(n) mod 1, is **that if the binary expansion of x(0) is finite, the system converges to 0**. This is obvious, because if we chop off one binary digit with each iteration, we eventually run out of binary digits, if the number of binary digits is finite.

Any fractional power of 2 will have a finite decimal expansion, and hence these values will converge to zero. (A fractional power of 2 is a fraction p/q where $q = 2^n$ for some n. For example, 3/256, which in binary is .00000011.)

In our first example, we started with x = .1 (where here .1 was a decimal value, or 1/10). Let's expand 1/10 into binary:

.00011001100110011001…

After the first 4 iterations, which chops off the leading 0001, we are left with a repeating 1001, 1001, 1001, etc. This is an infinite binary number, so it doesn't get any shorter as we chop off places. So we cycle through the same four numbers over and over. Namely, the four numbers corresponding to:

.10011001100110011001…
.00110011001100110011…
.01100110011001100110…
.11001100110011001100…

These correspond to the decimal numbers .6, .2, .4, .8, as we saw previously in the table.

All rational numbers (p/q, where q is not zero) that are not fractional powers of 2 (i.e., q is not a power of 2) will eventually cycle.

For example, the starting decimal x(0) = .71 begins to cycle after 21 iterations. We have x(22) = x(2) = .84, x(23) = x(3) = .68, and so on.

The starting decimal x(0) = .7182 begins to cycle after 502 iterations. We have x(503) = x(3) = .7456.

What if we start with an an irrational number, such as the square root of two, or π? Clearly these have non-repeating binary expansions, **so the system is chaotic if x(0) is irrational.**

Return to Part 2

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from *The Laissez Faire City Times*, Vol 3, No 24, June

14, 1999

Chaos and Fractals in Financial Markets

Part 3

by J. Orlin Grabbe

Hazardous World

Many things in life are random. They are governed by probability, by chance, by hazard, by accident, by the god Hermes, by fortune. So we measure them by probability—by our one-pound jar of jam.

Places where there is more jam are more likely to happen, but the next outcome is uncertain. The next outcome might be a low probability event. Or it might be a high probability event, but there may be more than one of these.

Radioactive decay is measured by probability. The timing of the spontaneous transformation of a nucleus (in which it emits radiation, loses electrons, or undergoes fission) cannot be predicted with any certainty.

Some people don't like this aspect of the world. They prefer to believe there are "hidden variables" which really determine radioactive decay, and if we only understood what these hidden variables were, it would all be precisely predictable, and we could return to the paradise of a Laplacian universe.

Well, if there are hidden variables, I sure wish someone would identify them. If wishes were horses, David Bohm would ride.[1] Albert Einstein liked to say, "God doesn't play dice." But if God wanted to play dice, he didn't need Albert Einstein's permission. It sounds to me like "hidden" is just another name for probability. "Was it an accident?" "No, it was caused by hidden forces." Hidden variable theorists all believe in conspiracy.

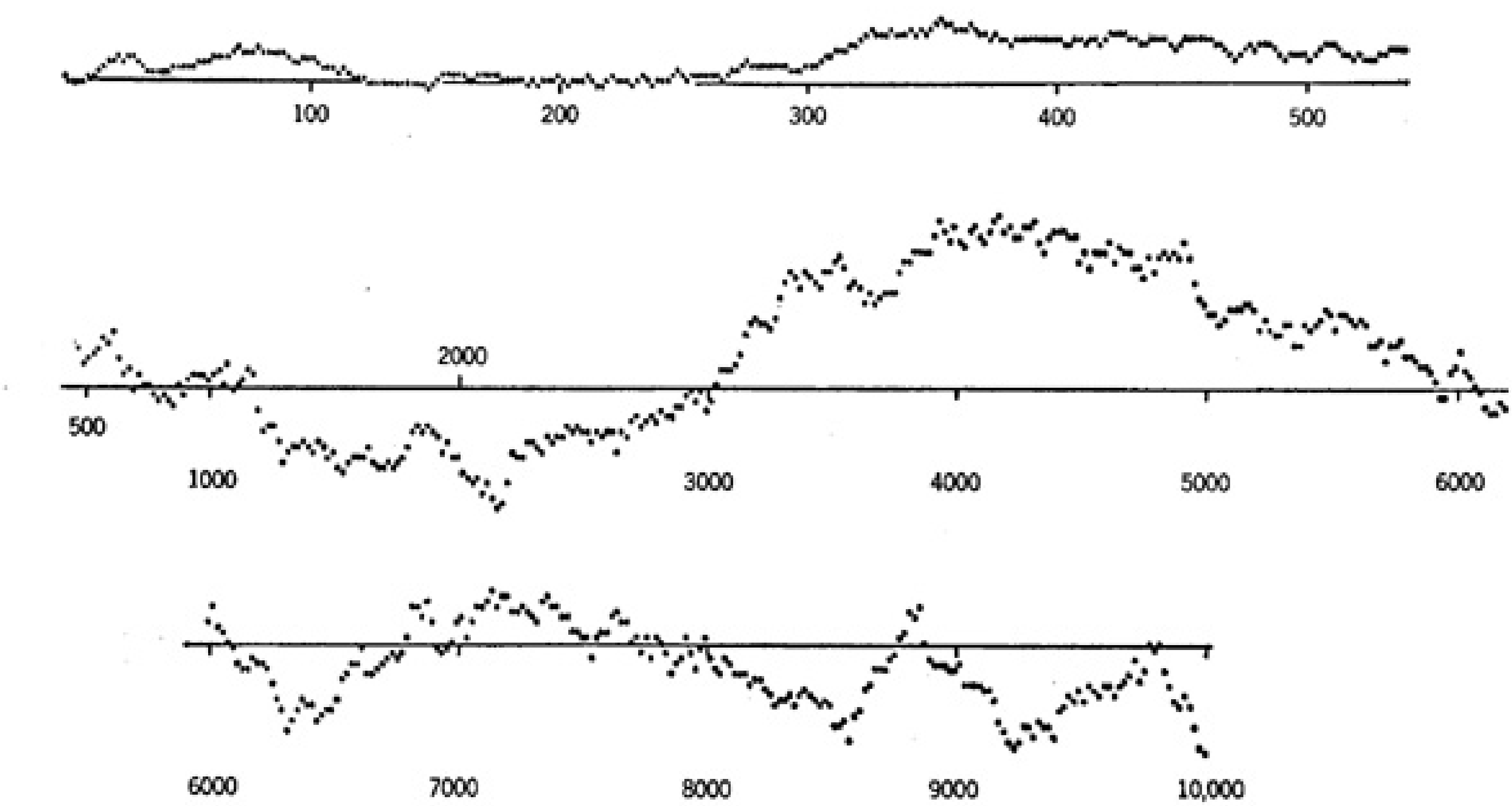
But, guess what? People who believe God doesn't play dice use probability theory just as much as everyone else. So, without further ado, let's return to our discussion of probability.

Coin Flips and Brownian Motion

We can create a kind of Brownian motion (or Bachelier process) by flipping coins. We start with a variable x = 0. We flip a coin. If the coin comes up heads, we add 1 to x. If the coin comes up tails, we subtract 1 from x. If we denote the input x as x(n) and the output x as x(n+1), we get a dynamical system:

$x(n+1) = x(n) + 1$, with probability $p = ½$
$x(n+1) = x(n) – 1$, with probability $q = ½$.

Here **n** represents the current number of the coin flip, and is our measure of time. So to create a graph of this system, we put **n** (time) on the horizontal axis, and the variable x(n) on the vertical axis. This gives a graph of a very simple type of Brownian motion (a *random walk*), as seen in the graphic below. At any point in time (at any value of **n**), the variable x(n) represents the total number of heads minus the total number of tails. Here is one picture of 10,000 coin flips:



Much of finance is based on a simple probability model like this one. Later we will change this model by changing the way we measure probability,

A Simple Stochastic Fractal

Using probability, it is easy to create fractals. For example, here is a dynamical system which creates a Simple Stochastic Fractal. The system has two variables, x and y, as inputs and outputs:

$x(n+1) = - y(n)$
$y(n+1) = x(n)$

with probability p = ½ , but

$x(n+1) = 1 + 2.8*(x(n)-1)/(x(n)*x(n)-2*x(n)+2+y(n)*y(n))$
$y(n+1) = 2.8*y(n)/(x(n)*x(n)-2*x(n)+2+y(n)*y(n))$

with probability q = ½.

We map x and y on a graph of two dimensions. If the coin flip comes up heads, we iterate the system by the first two equations. This iteration represents a simple 90-degree rotation about the origin (0,0). If the coin flip comes up tails, we iterate the system by the second two equations. This second type of iteration contracts or expands the current point with respect to (1,0).

To see this Simple Stochastic Fractal system works in real time, be sure Java is enabled on your web browser, and click here. [2]

Simple stochastic dynamical systems create simple fractals, like those we see in nature and in financial markets. But in order to get from Bachelier to Mandelbrot, which requires a change in the *way we measure probability*, it will be useful for us to first think about something simpler, such as the *way we measure length*.

One we've learned to measure length, we'll find that probability is jam on toast.

Sierpinski and Cantor Revisited

In Part 2, when we looked at Sierpinski carpet, we noted that a Sierpinski carpet has a Hausdorff dimension D = log 8/log 3 = 1.8927… So if we have a Sierpinski carpet with length 10 on each side, we get

$N = r^D = 10^D = 10^{1.8927} = 78.12$

smaller copies of the original. (For a nice round number, we can take 9 feet on a side, and get $N = 9^{1.8927} = 64$ smaller copies.) Since each of these smaller copies has a length of one foot on each side, we can call these "square feet". But really they are "square Sierpinskis", because Sierpinski carpet is not like ordinary carpet.

So let's ask the question: How much space (area) does Sierpinski carpet take up relative to ordinary carpet? We have 78.12 smaller copies of the original. So if we know how much area (in terms of ordinary carpet) each of these smaller copies takes up, we can multiply that number by 78.12 and get the answer.

Hmmm. To calculate an answer this question, let's take the same approach we did with Cantor dust. In the case of Cantor dust, we took a line of length one and began cutting holes in it. We divided it into three parts and cut out the middle third, like this:

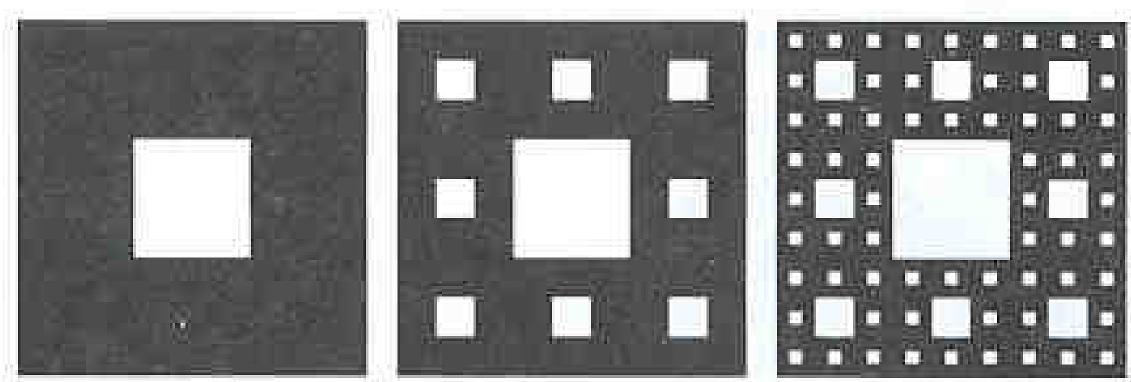
```
0__________________________________________________1

0________________1/3               2/3_______________1
```

That left 2/3 of the original length. Then we cut out the middle thirds of each of the two remaining lines, which left 2/3 of what was there before; that is, it left (2/3)(2/3), or $(2/3)^2$. And so on. After the **n**-th step of cutting out middle thirds, the length of the remaining line is $(2/3)^n$.

If we take the limit as **n** $\to \infty$ (as **n** goes to infinity), we have $(2/3)^n \to 0$ (that is, we keep multiplying the remaining length by 2/3, and by so doing, we eventually reduce the remaining length to zero). [3] So Cantor dust has a length of zero. What is left is an infinite number of disconnected points, each with zero dimension. So we said Cantor dust had a topological dimension of zero. Even though we started out with a line segment of length one (with a dimension of one), before we began cutting holes in it.

Well. Now let's do the same thing with Sierpinski carpet. We have an ordinary square and divide the sides into three parts (divide by a scale factor of 3), making 9 smaller squares. Then we throw out the middle square, leaving 8 smaller squares, as in the figure below:



So we have left 8/9 of the original area. Next, we divide up each of the smaller squares and throw out the centers. Each of them now has 8/9 of its original area, so the area of the big square has been reduced to (8/9)(8/9) of its original size, or to $(8/9)^2$. At the **n**-th step of this process, we have left $(8/9)^n$ of the original area. Taking the limit as $\mathbf{n} \to \infty$ (as **n** goes to infinity), we have $(8/9)^n \to 0$. So the Sierpinski carpet has an area of zero.

What? This seems properly outrageous. The 78.12 smaller copies of the original Sierpinski carpet that measured 10 x 10 (or 64 smaller copies of an original Sierpinski carpet that measured 9 x 9), actually take up zero area. By this argument, at least. By this way of measuring things.

We can see what is happening, if we look at the Sierpinski carpet construction again. Note in the graphic above that the *outside* perimeter of the original big square never acquires any holes as we create the Sierpinski carpet. So the outside perimeter forms a loop: a closed line in the shape of a square. A loop of one dimension.

Next note that the border of the first *center* square we remove also remains intact. This leaves a second smaller (square) loop: a second closed line of one dimension, inside the original loop. Next, the centers of the 8 smaller squares also form even smaller (square) loops. If we continue this process forever, then in the limit we are left with an infinite number of disconnected loops, each of which is a line of one dimension. This is the Sierpinski carpet.

Now, with respect to Cantor dust, we said we had an infinite number of disconnected points, each with *zero* dimension, and then chose to say that Cantor dust itself had a *topological dimension of zero*. To be consistent, then, we must say with respect to the Sierpinski carpet, which is made up of an infinite number of disconnected loops, each of *one* dimension, that it has a *topological dimension of one*.

Hmm. Your eyebrows raise. Previously, in Part 2, I said Sierpinski carpet had an ordinary (or topological) dimension of 2 . That was because we started with a 10 by 10 square room we wanted to cover with carpet. So, intuitively, the dimension we were working in was 2.

The confusion lies in the phrase "topological or ordinary" dimension. These are not the same. Or, better, we need more precision. In the case of Sierpinski carpet, we started in a context of two-dimensional floor space. Let's call this a *Euclidean dimension of 2*. It corresponds to our intuitive notion that by covering a floor space with carpet, we are doing things in a plane of 2 dimensions. But, once we measure all the holes in the carpet, we discovered that what we are left with is carpet that has been entirely consumed by holes. It has zero area. What is left over is an infinite number of disconnected closed loops, each of which has a dimension of one. So, in this respect, let's say that Sierpinski carpet has a topological dimension of one.

Thus we now have three different dimensions for Sierpinski carpet: a Euclidean dimension (E) of 2, a topological dimension (T) of 1, and a Hausdorff dimension (D) of 1.8927…

Similarly, to create Cantor dust, we start with a line of one dimension. Our *working space* is one dimension. So let's say Cantor dust has a Euclidean dimension (E) of 1, a topological dimension (T) of 0, and a Hausdorff dimension (D) of log 2/log 3 = .6309…

So here are three different ways [4] of looking at the same thing: the Euclidean dimension (E), the topological dimension (T), and the Hausdorff dimension (D). Which way is best?

Blob Measures Are No Good

Somewhere (I can't find the reference) I read about a primitive tribe that had a counting system that went: 1, 2, 3, many. There were no names for numbers beyond 3. Anything numbered beyond three was referred to as "many".

"We're being invaded by foreigners!" "How many of them are there?" "Many!"

It's not a very good number system, since it can't distinguish between an invading force of five and an invading force of fifty.

(Of course, if the enemy was in sight, one could get around the lack of numbers. Each individual from the local tribe could pair himself with a invader, until there were no unpaired invaders left, and the result would be an opposing force that matched in number the invading force. George Cantor, the troublemaker who invented set theory, would call this a *one-to-one correspondence*.)

"Many." A blob. Two other blob measures are: *zero* and *infinity*. For example, Sierpinski carpet has zero area and so does Cantor dust. But they are not the same thing.

We get a little more information if we know that Cantor dust has a topological dimension of zero, while a Sierpinski carpet has a topological dimension of one. But topology often conceals more than it reveals. The topological dimension of *zero* doesn't tell us how Cantor dust differs from a single point. The topological dimension of *one* doesn't tell us how a Sierpinski carpet differs from a circle.

If we have a circle, for example, it is fairly easy to measure its length. In fact, we can just measure the radius **r** and use the formula that the length L (or "circumference" C) is

$L = C = 2 \pi r$

where $\pi = 3.141592653\ldots$ is known accurately to millions of decimal places. But suppose we attempt to measure the length of a Sierpinski carpet? After all, we just said a Sierpinski carpet has topological dimension of one, like a *line*, so how long is it? What is the length of this here Sierpinski carpet compared to the length of that there circle?

To measure the Sierpinski carpet we began measuring smaller and smaller squares, so we keep having to make our measuring rod smaller and smaller. But as the squares get smaller, there are more and more of them. If we actually try to do the measurement, we discover the length goes to infinity. (I've measured my Sierpinski carpet; haven't you measured yours yet?)

Infinity. A blob. "How long is it?" "Many!"

Coastlines and Koch Curves

If you look in the official surveys of the length of borders between countries, such as that between Spain and Portugal, or between Belgium and The Netherlands, you will find they can differ by as much as 20 percent. [5]

Why is this? Because they used measuring rods that were of different lengths. Consider: one way to measure the length of something is to take a measuring rod of length **m,** lay it alongside what you are measuring, mark the end point of the measuring rod, and repeat the process until you have the number N of measuring rod lengths. Then for the total length L of the object, you have

L = m N

(where "m N" means "m times N").

For example, suppose we are measuring things in feet, and we have a yardstick (**m** = 3). We lay the yardstick down the side of a football field, and come up with N = 100 yardstick lengths. So the total length is

L = 3 (100) = 300 feet.

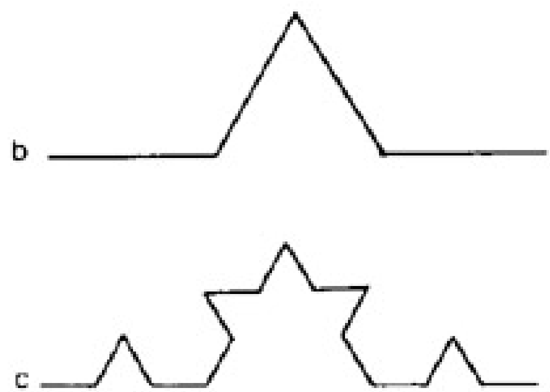
And, if instead of using a yardstick, we used a smaller measuring rod—say a ruler that is one foot long, we would still get the same answer. Using the ruler, **m** = 1 and N = 300, so L = 1 (300) = 300 feet. This may work for the side of a football field, but does it work for the coastline of Britain? Does it work for the border length between Spain and Portugal?

Portugal is a smaller country than Spain, so naturally it used a measuring rod of shorter length. And it came up with an estimate of the length of the mutual border that was longer than Spain's estimate.

We can see why if we imagine measuring, say, the coastline of Britain. If we take out a map, lay a string around the west coast of Britain, and then multiply it by the map scale, we'll get an estimate of the "length" of the western coastline. But if we come down from our satellite view and actually visit the coast in person, then we will see that there are a lot of ins and outs and crooked jags in the area where the ocean meets the land. The smaller the measuring rod we use, the longer will our measure become, because we capture more of the length of the irregularities. The difference between a coastline and the side of a football field is the coastline is fractal and the side of the football field isn't.

To see the principles involved, let's play with something called a Koch curve. First we will construct it. Then we will measure its length. You can think of a Koch curve as being as being a section of coastline.

We take a line segment. For future reference, let's say its length L is L = 1. Now we divide it into three parts (each of length 1/3), and remove the middle third. But we replace the middle third with *two* line segments (each of length 1/3), which can be thought of as the other two sides of an equilateral triangle. This is stage two (b) of the construction in the graphic below:



At this point we have 4 smaller segments, each of length 1/3, so the total length is 4(1/3) = 4/3. Next we repeat this process for each of the 4 smaller line segments. This is stage three (c) in the graphic above. This gives us 16 *even smaller* line segments, each of length 1/9. So the total length is now 16/9 or $(4/3)^2$.

At the **n**-th stage the length is $(4/3)^n$, so as **n** goes to infinity, so does the length L of the curve. The final result "at infinity" is called a Koch curve. At each of its points it has a sharp angle. Just like, say, Brownian motion seen at smaller and smaller intervals of time. (If we were doing calculus, we would note there is no tangent at any point, so the Koch curve has no derivative. The same applies to the path of Brownian motion.)

However, the Koch curve is continuous, because we can imagine taking a pencil and tracing its (infinite) length from one end to the other. So, from the topological point of view, the Koch curve has a dimension of one, just like the original line. Or, as a topologist would put it, we can deform (stretch) the original line segment into a Koch curve without tearing or breaking the original line at any point, so the result is still a "line", and has a topological dimension T = 1.

To calculate a Hausdorff dimension, we note that at each stage of the construction, we replace each line segment with N = 4 segments, after dividing the original line segment by a scale factor r = 3. So its Hausdorff dimension D = log 4/log 3 = 1.2618…

Finally, when we constructed the Koch curve, we did so by viewing it in a Euclidean plane of two dimensions. (We imagined replacing each middle line segment with the other two sides of an equilateral triangle—which is a figure of 2 dimensions.) So our working space is the Euclidean dimension E = 2.

But here is the key point: as our measuring rod got smaller and smaller (through repeated divisions by 3), the measured length of the line got larger and larger. Just like a coastline. (And just like the path of Brownian motion.) The total length $(4/3)^n$ went to infinity as **n** went to infinity. At the **n-**th stage of construction we had $N = 4^n$ line segments, each of length **m** = $(1/3)^n$, so the total length L was:

$L = m\ N = (1/3)^n\ 4^n = (4/3)^n$.

Well, there’s something wrong with measuring length this way. Because it gives us a blob measure. Infinity. "Many."

Which is longer, the coast of Britain or the coast of France? Can’t say. They are both infinity. Or maybe they have the same length: namely, infinity. They are both "many" long. Well, how long is the coastline of Maui? Exactly the same. Infinity. Maui is many long too. (Do you feel like a primitive tribe trying to count yet?)

Using a Hausdorff Measure

The problem lies in our measuring rod **m**. We need to do something to fix the problem that as **m** gets smaller, the length L gets longer. Let’s try something. Instead of

$L = m\ N$,

let’s adjust **m** by raising it to some power **d**. That is, replace **m** by $\mathbf{m^d}$:

$L = m^d\ N$.

This changes our way of measuring length L, because only when **d** = 1 do we get the same measure of length as previously.

If we do this, replace **m** by $\mathbf{m^d}$, we discover that for values of **d** that are too small, **L** still goes to infinity. For values of **d** that are too large, **L** goes to zero. Blob measures. There is only one value of **d** that is just right: namely, the Hausdorff dimension **d** = D. So our measure of length becomes:

$L = m^D N$

How does this work for the Koch curve? We saw that for a Koch curve the number of line segments at stage **n** was $N = 4^n$, while the length of a line segment $\mathbf{m} = (1/3)^n$. So we get as our new measure of the length L of a Koch curve (where D = log 4/log 3):

$L = m^D N = ((1/3)^n)^D (4^n) = ((1/3)^n)^{\log 4/\log 3} (4^n) = 4^{-n} (4^n) = 1.$

Success. **We've gotten rid of the blob**. The length L of the Koch curve under this measure turns out to be the length of the original line segment. Namely, L = 1.

The Hausdorff dimension D is a natural measure associated with our measuring rod **m**. If we are measuring a football field, then letting D = 1 works just fine to measure out 100 yards. But if we are dealing with Koch curves or coastlines, then some other value of D avoids the futile exercise having the measured length fully dependent on the length of the measuring rod.

To make sure we understand how this works, let's calculate the length of a Sierpinski carpet constructed from a square with a starting length of 1 on each side. For the Sierpinski carpet, N gets multiplied by 8 at each stage, while the measure rod gets divided by 3. So the length at stage **n** is:

$L = m^D N = ((1/3)^n)^D (8^n) = ((1/3)^n)^{\log 8/\log 3} (8^n) = 8^{-n} (8^n) = 1.$

Hey! We've just destroyed the blob again! We have a finite length. It's not zero and it's not infinity. Under this measure, as we go from the original square to the ultimate Sierpinski carpet, the length stays the same. The *Hausdorff length (area)* of a Sierpinski carpet is 1, assuming that we started with a square that was 1 on each side. (We can informally choose to say that the "area" covered by the Sierpinski carpet is "one square Sierpinski", because we need a Euclidean square, the length of each side of which is 1, in order to do the construction.) [6]

[Note that if we use a **d** > D, such as **d** = 2, then the length L of the Sierpinski carpet goes to zero, as **n** goes to infinity. And if we use a **d** < D, such as **d** = 1, then the length goes to infinity, as **n** goes to infinity. So, doing calculations using the Euclidean dimension E =2 leads to an "area" of zero, while calculations using the topological dimension T=1 leads to a "length" of infinity. Blob measures.]

If instead we have a Sierpinski carpet that is 9 on each side, then to calculate the "area", we note that the number of Sierpinski copies of the initial square which has a side of length 1 is (dividing each side into r = 9 parts) $N = r^D = 9^D = 64$. Thus, ***using the number of Sierpinski squares with a side of length 1, then, as the basis for our measuresment***, the Sierpinski carpet with 9 on each side has an "area" of $N = 9^D = 9^{1.8927\ldots} = 64$. A Sierpinski carpet with 10 on each side has an "area" of $N = 10^{1.8927\ldots} = 78.12$. And so on.

The Hausdorff dimension, D = 1.8927…, is closer to 2 than to 1, so having an "area" of 78.12 (which is in the region of $10^2 = 100$) for a side length of 10 is more esthetically pleasing than saying the "area" is zero.

This way of looking at things lets us avoid having to say of two Sierpinski carpets (one of side 9 and the other of side 1): "Oh, they're exactly the same. They both have *zero* area. They both have *infinite* length!" Blah, blah, **blob, blob**.

Indeed do "many" things come to pass.

To see a Sierpinski Carpet Fractal created in real time, using probability, be sure Java is enabled on your web browser, and click here.

Jam Session

One of the important points of the discussion above is that the *power* (referring specifically to the Hausdorff D) to which we raise things is crucial to the resulting measurement. If we "square" things (raise them to the power 2) at times when 2 is not appropriate, we get blob measures equivalent to, say, "this regression coefficient is ‘many’ ".

Unfortunately, people who measure things using the wrong dimension often think they are saying something other than "many." They think their measurements mean something. They are self-deluded. Many empirical and other results in finance are an exercise in self-delusion, because the wrong dimension has been used in the calculations.

When Louis Bachelier gave the first mathematical description of Brownian motion in 1900, he said the probability of the price distribution changes with the square root of time. We modified this to say that the probability of the *log of the price* destribution changes with the square root of time—and **from now on, without further discussion, we will pretend that that’s what Bachelier said also**.

The issue we want to consider is whether the appropriate dimension for time is D = ½. In order to calculate probability should we use $T^{1/2}$, or T^D, where D may take values different from ½?

This was what Mandelbrot was talking about when he said the empirical distribution of price changes was "too peaked" to come from a normal distribution. Because the dimension D = ½ is only appropriate in the context of a normal distribution, which arises from simple Brownian motion.

We will explore this issue in Part 4.

Notes

[1] David Bohm’s hidden-variable interpretation of the quantum pilot wave (which obeys the rules of quantum probability) is discussed in John Gribbin, *Schrodinger’s Kittens and the Search for Reality*, Little, Brown and Company, New York, 1995.

[2] If your computer monitor has much greater precision than assumed here, you can see much more of the fractal detail by using a larger area than 400 pixels by 400 pixels. Just replace "200" in the Java program by one-half of your larger pixel width, and recompile the

applet.

[3] Note that in Part 2, we measured the length of the line segments that we *cut out*. Here, however, we are measuring the length of the line segment that is *left behind*. Both arguments, of course, lead to the same conclusion. We cut out a total of length one from the original line of length one, leaving behind a segment of length zero.

[4] This three-fold classification corresponds to that in Benoit B. Mandelbrot, *The Fractal Geometry of Nature*, W.H. Freeman and Company, New York, 1983.

[5] L. F. Richardson, "The problem of contiguity: an appendix of statistics of deadly quarrels," *General Systems Yearbook*, 6, 139-187, 1961.

[6] Whether one refers to the resulting carpet as "1 square Sierpinski" or just "1 Sierpinski" or just "a carpet with a side length of 1" is basically a matter of taste and semantic convenience.

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from *The Laissez Faire City Times*, Vol 3, No 26, June 28, 1999

A Simple Stochastic Fractal

If you can read this, then your browser is not set up for Java. . .

This applet generates a simple stochastic fractal. It maps points (x,y) on a plane. You can think of the center of the applet screen as being (0,0). It plots 150,000 points in an area that covers 400 pixels x 400 pixels on your computer screen (i.e. a total of 160,000 pixels).

First a random number ("**ran**") between 0 and 1 is selected. If **ran** is greater than or equal to 1/2, then we rotate the current (x,y) point 90 degress to the left, and plot a new point. If **ran** is less than 1/2, then, depending on the distance of (x,y) from (1,0), we expand or contract the values of x and y with respect to (1,0).

Here is the java source code.

Sierpinski Carpet Fractal

If you can read this, then your browser is not set up for Java. . .

This applet generates a Sierpinski carpet. If you look at the Java source code you will see that it generates a random number, and then with probability 1/8 does the following:

- divides the current (x,y) value by 3,
- and may also add either 1/3 or 2/3 to x, or to y, or to both.

The key omission, in order to create the Sierpinski carpet holes, is that zero probability is assigned to the possibility

```
Xnew = X/3 + .3333333333333333;
Ynew = Y/3 + .3333333333333333;
```

This omission creates a hole at each smaller level of the carpet.

Here is the java source code.

Chaos and Fractals in Financial Markets

Part 4

by J. Orlin Grabbe

Gamblers, Zero Sets, and Fractal Mountains

Henry and Thomas are flipping a fair coin and betting $1 on the outcome. If the coin comes up heads, Henry wins a dollar from Thomas. If the coin comes up tails, Thomas wins a dollar from Henry. Henry's net winnings in dollars, then, are the total number of heads minus the total number of tails.

But we saw all this before, in Part 3. If we let x(n) denote Henry's net winnings, then x(n) is determined by the dynamical system:

x(n) = x(n) + 1, with probability p = ½
x(n) = x(n) – 1, with probability q = ½.

The graph of 10,000 coin tosses in Part 3 simply shows the fluctuations in Henry's wealth (starting from 0) over the course of the 10,000 coin tosses.

Let's do this in real time, although we will restrict ourselves to 3200 coin tosses. Let's plot Henry's winnings for a new game that lasts for 3200 flips of the coin. You can quickly see the results of many games with a few clicks of your mouse. Make sure Java is enabled on your web browser, and click here.

There are three things to note about this demonstration:

1. Even though the odds are even for each coin clip, winnings or losses can at times add up significantly. Even though a head or a tail is equally probable for each coin flip, there can be a series of "runs" that result in a large loss to either Henry or Thomas. This fact is important in understanding the "gambler's ruin" problem discussed later.
2. The set of points where x(n) comes back to x(n) = 0 (that is, the points where wins and losses are equalized), is called the **zero set** of the system. Using **n** as our measure of time, the time intervals between each point of the zero set are independent, but form clusters, much like Cantor dust. To see the zero set plotted for the coin tossing game, make sure Java is enabled on your web browser and click here. The zero set represents those times at which Henry has broken even. (Make sure to run the series of coin flips multiple times, to observe various patterns of the zero set.)
3. The fluctuations in Henry's winnings form an outline that is suggestive of mountains and valleys. In fact, this is a type of "Brownian landscapes" that we see around us all the time. To create different "alien" landscapes, for, say, set

decorations in a science fiction movie, we can change the probabilities. The effects in three dimensions, with a little color shading, can be stunning.

Since we will later be discussing motions that are not Brownian, and distributions that are not normal (not Gaussian), it is important to first point out an aspect of all this that is somewhat independent of the probability distribution. It's called the Gambler's Ruin Problem. You don't need nonnormal distributions to encounter gambler's ruin. Normal ones will do just fine.

Futures Trading and the Gambler's Ruin Problem

This section explains how casinos make most of their money, as well as why the traders at Goldman Sachs make more money speculating than you do. It's not necessarily because they are smarter than you. It's because they have more money. (However, we will show how the well-heeled can easily lose this advantage.)

Many people assume that the futures price of a stock index, bond, foreign currency, or commodity like gold represents a fair bet. That is, they assume that the probability of an upward movement in the futures price is equal to the probability of a downward movement, and hence the mathematical expectation of a gain or loss is zero. They use the analogy of flipping a fair coin. If you bet $1 on the outcome of the flip, the probability of your winning $1 is one-half, while the probability of losing $1 is also one-half. Your expected gain or loss is zero. For the same reason, they conclude, futures gains and futures losses will tend to offset each other in the long run.

There is a hidden fallacy in such reasoning. Taking open positions in futures contracts is not analogous to a single flip of a coin. Rather, the correct analogy is that of a ***repeated series of coin flips with a stochastic termination point***. Why? Because of limited capital. Suppose you are flipping a coin with a friend and betting $1 on the outcome of each flip. At some point either you or your friend will have a run of bad luck and will lose several dollars in succession. If one player has run out of money, the game will come to an end. The same is true in the futures market. If you have a string of losses on a futures position, you will have to post more margin. If at some point you cannot post the required margin, you will have to close out the contract. You are forced out of the game, and thus you cannot win back what you lost. In a similar way, in 1974, Franklin National and Bankhaus I. D. Herstatt had a string of losses on their interbank foreign exchange trading positions. They did not break even in the long run because there was no long run. They went broke in the intermediate run. This phenomenon is referred to in probability theory as the ***gambler's ruin problem*** [1].

What is a "fair" bet when viewed as a single flip of the coin, is, when viewed as a series of flips with a stochastic ending point, really a different game entirely whose odds are quite different. The probabilities of the game then depend on the relative amounts of capital held by the different players.

Suppose we consider a betting process in which you will win $1 with probability p

and lose $1 with probability q (where $q = 1 - p$). You start off with an amount of $$W$. If your money drops to zero, the game stops. Your betting partner—the person on the other side of your bet who wins when you lose and loses when you win—has an amount of money $$R$. What is the probability you will eventually lose all of your wealth W, given p and R? From probability theory [1] the answer is:

$$\text{Ruin probability} = \frac{(q/p)^{W+R} - (q/p)^{W}}{(q/p)^{W+R} - 1}, \qquad \text{for } p \neq q$$

$$\text{Ruin probability} = 1 - [W/(W + R)], \qquad \text{for } p = q = .5.$$

An Example

You have $10 and your friend has $100. You flip a fair coin. If heads comes up, he pays you $1. If tails comes up, you pay him $1. The game ends when either player runs out of money. What is the probability your friend will end up with all of your money? From the second equation above, we have $p = q = .5$, $W = \$10$, and $R = \$100$. Thus the probability of your losing everything is:

1 - (10/(10 + 100)) =.909.

You will lose all of your money with 91 percent probability in this supposedly "fair" game.

Now you know how casinos make money. Their bank account is bigger than yours. Eventually you will have a losing streak, and then you will have to stop playing (since the casinos will not loan you infinite capital).

The gambler’s ruin odds are the important ones. True, the odds are stacked against the player in each casino game: heavily against the player for kino, moderately against the player for slots, marginally against the player for blackjack and craps. (Rules such as "you can only double down on 10s and 11s" in blackjack are intended to turn the odds against the player, as are the use of multiple card decks, etc.) But the chief source of casino winnings is that people have to stop playing once they’ve had a sufficiently large losing streak, which is inevitable. (Lots of "free" drinks served to the players help out in this process. From the casino’s point of view, the investment in free drinks plays off splendidly.)

Note here that "wealth" (**W** or **R** in the equation) is defined as the number of betting units: $1 in the example. The more betting units you have, the less probability there is you will be hit with the gambler’s ruin problem. So you if you sit at the blackjack table at Harrah’s with a $1000 minimum bet, you will need to have 100 times the total betting capital of someone who sits at the $10 minimum tables, in order to have the same odds vis-à-vis the dealer.

A person who has $1000 in capital and bets $10 at a time has a total of W = 1000/10 = 100 betting units. That's a fairly good ratio.

While a person who has $10,000 in capital and bets $1000 at a time has W = 10000/1000 = 10 betting units. That's lousy odds, no matter the game. It's loser odds.

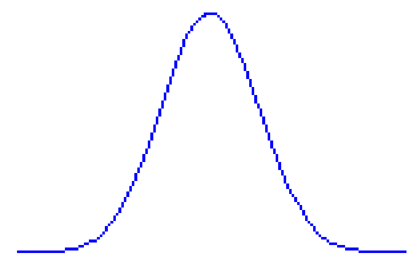
Gauss vs. Cauchy

We measure probability with our one-pound jar of jam. We can distribute the jam in any way we wish. If we put it all at the point x = 5, then we say "x = 5 with certainty" or "x = 5 with probability 1."

Sometimes the way the jam is distributed is determined by a simple function. The *normal* or *Gaussian* distribution distributes the jam (probability) across the real line (from minus infinity to plus infinity) using the *density* function:

$f(x) = [1/(2\pi)^{0.5}] \exp(-x^2/2)$, $-\infty < x < \infty$

Here f(x) creates the nice bell-shaped curve we have seen before (**x** is on the horizontal line, and f(x) is the blue curve above it):



The jam (probability) is smeared between the horizontal line and the curve, so the *height of the curve* at each point (given by f(x)) indicates that point's probability relative to some other point. The curve f(x) is called the *probability density*.

So we can calculate the probability density for each value of x using the function f(x). Here are some values:

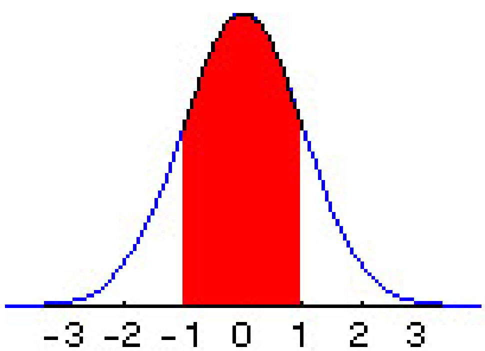
x	f(x)
-3	.0044
-2	.0540
-1	.2420
-.75	.3011
-.50	.3521

-.25	.3867
0	.3989
.25	.3867
.50	.3521
.75	.3011
1	.2420
2	.0540
3	.0044

At the center value of x = 0, the probability density is highest, and has a value of f(x) = .3989. Around 0, the probability density is spread out symmetrically in each direction.

The entire one-pound jar of jam is smeared underneath the curve between – ∞ and + ∞ . So the total probability, the *total area under the curve, is 1*. In calculus the area under the curve is written as an integral, and since the total probability is one, the integral from **-** ∞ to + ∞ of the jam-spreading function f(x) is 1.

The probability that **x** lies between **a** and **b**, a < x < b, is just the area under the curve (the amount of jam) measured from **a** to **b**, as indicated by the red portion in the graphic below, where **a** = -1 and **b** = +1:



Instead of writing this integral in the usual mathematical fashion, which requires using a graphic in the *html* world of your web browser, I will simply denote the integral from **a** to **b** of f(x) as:

I(a,b) f(x) dx.

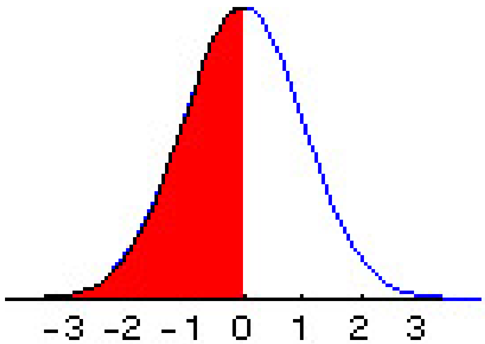
I(a,b) f(x) dx, then, is the area under the f(x) curve from **a** to **b**. In the graphic above, we see pictured I(-1,1). And since the total probability (total area under the curve) across all values of **x** (from - ∞ to + ∞) is 1, we have

I(- ∞ ,∞) f(x) dx = 1.

A little more notation will be useful. We want a shorthand way of expressing the probability that x < b. But the probability that x < b is the same as the probability that -∞ < x < b. So this value is given by the area under the curve from -∞ to b. We will write this as F(b):

F(b) = I(-∞ ,b) f(x) dx = area under curve from minus infinity to b.

Here is a picture of F(b) when b = 0:



For any value **x**, F(x) is the *cumulative* probability function. It represents the total probability up to (and including) point **x**. It represents the probability of all values smaller than (or equal to) **x**.

(Note that since the *area* under the curve at a *single point* is zero, whether we include the point **x** itself in the cumulative probability function F(x), or whether we only include all points less than **x**, does not change the value of F(x). However, our understanding will be that the point **x** itself is included in the calculation of F(x).)

F(x) takes values between 0 and 1, corresponding to our one-pound jar of jam. Hence

F(-∞) = 0, while

F(+∞) = 1.

The probability between **a** and **b**, a <x < b, then, can be written simply as

F(b) – F(a).

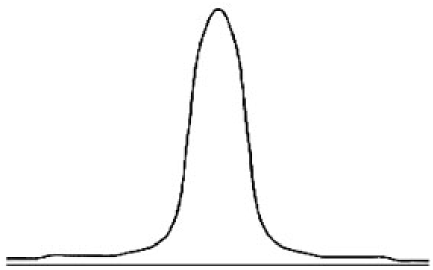
The probability x > **b** can be written as:

1 – F(b).

Now. Here is a different function for spreading probability, called the ***Cauchy density***:

$g(x) = 1/[\pi (1 + x^2)], - \infty < x < \infty$

Here is a picture of the resulting Cauchy curve:



It it nice and symmetric like the normal distribution, but is relatively more concentrated around the center, and taller in the tails than the normal distribution. We can see this more clearly by looking at the values for g(x):

x	g(x)
-3	.0318
-2	.0637
-1	.1592
-.75	.2037
-.50	.2546
-.25	.2996
0	.3183
.25	.2996
.50	.2546
.75	.2037
1	.1592
2	.0637
3	.0318

At every value of **x**, the Cauchy density is lower than the normal density, until we get out into the extreme tails, such as 2 or 3 (+ or -).

Note that at –3, for example, the probability density of the Cauchy distribution is g(-3) = .0318, while for the normal distribution, the value is f(-3) = .0044. There is more than 7 times as much probability for this extreme value with the Cauchy distribution than there is with the normal distribution! (The calculation is .0318/.0044 = 7.2.) Relative to the normal, the Cauchy distribution is *fat-tailed*.

To see a more detailed plot of the normal density minus the Cauchy density, make sure Java is enabled on your web browser and click here.

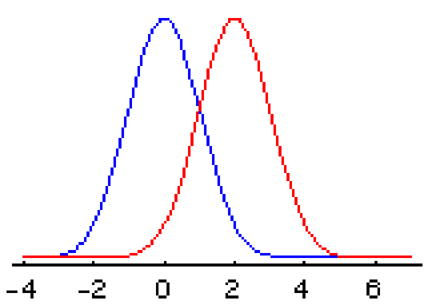
As we will see later, there are other distributions that have more probability in the tails than the normal, and *also* more probability at the peak (in this case, around 0). But since the total probability must add up to 1, there is, of course, less probability in the intermediate ranges. Such distributions are called *leptokurtic*. Leptokurtic distributions have more probability both in the tails and in the center than does the normal distribution, and are to be found in all asset markets—in foreign exchange, shares of stock, interest rates, and commodity prices. (People who pretend that the empirical distributions of changes in log prices in these markets are *normal*, rather than *leptokurtic*, are sadly deceiving themselves.)

Location and Scale

So far, as we have looked at the normal and the Cauchy densities, we have seen they are centered around zero. However, since the density is defined for all values of **x**, $-\infty < x < \infty$, the center can be elsewhere. To move the center from zero to a **location m**, we write the normal probability density as:

$f(x) = [1/(2\pi)^{0.5}] \exp(-(\mathbf{x}-\mathbf{m})^2/2)$, $-\infty < x < \infty$.

Here is a picture of the normal distribution after the location has been moved from **m** = 0 (the blue curve) to **m** = 2 (the red curve):



For the Cauchy density, the corresponding alteration to include a location parameter **m** is:

$g(x) = 1/[\pi (1 + (\mathbf{x}-\mathbf{m})^2)]$, $-\infty < x < \infty$

In each case, the distribution is now centered at **m**, instead of at 0. Note that I say **"location paramter m"** and ***not*** "mean **m**". The reason is simple. For the Cauchy distribution, a mean doesn't exist. But a location parameter, which shows where the probability distribution is centered, does.

For the normal distribution, the location parameter **m** is the same as the mean of the distribution. Thus a lot of people who are only familiar with the normal distribution confuse the two. They are not the same.

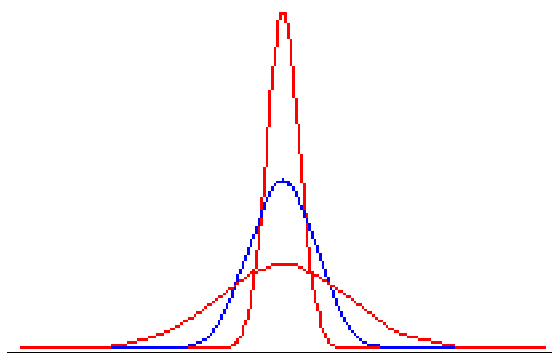
Similarly, for the Cauchy distribution the standard deviation (or the variance, which is the square of the standard deviation) doesn't exist. But there is a **scale parameter c** that ***shows how far you have to move in each direction from the location parameter m, in order for the area under the curve to correspond to a given probability***. For the normal distribution, the scale parameter **c** corresponds to the standard deviation. But a scale parameter **c** is defined for the Cauchy and for other, leptokurtic distributions for which the variance and standard deviation don't exist ("are infinite").

Here is the normal density written with the addition of a scale parameter **c**:

$f(x) = [1/(\mathbf{c}\ (2\pi)^{0.5})]\ \exp(-((\mathbf{x-m})/\mathbf{c})^2/2)\ ,\ -\infty < x < \infty$.

We divide (**x-m**) by **c**, and also multiply the entire density function by the reciprocal of **c**.

Here is a picture of the normal distribution for difference values of **c**:



The blue curve represents **c** = 1, while the peaked red curves has **c** < 1, and the flattened red curve has **c** > 1.

For the Cauchy density, the addition of a scale parameter gives us:

$g(x) = 1/[\mathbf{c}\pi\ (1 + ((\mathbf{x-m})/\mathbf{c})^2)],\ -\infty < x < \infty$

Just as we did with the normal distribution, we divide (**x-m**) by **c**, and also multiply the entire density by the reciprocal of **c**.

Operations with *location* and *scale* are well-defined, whether or not the *mean* or the *variance* exist.

Most of the probability distributions we are interested in in finance lie somewhere between the normal and the Cauchy. These two distributions form the "boundaries", so to speak, of our main area of interest. Just as the Sierpinski carpet has a Hausdorff dimension that is a fraction which is greater than its topological dimension of 1, but less than its Euclidean dimension of 2, so do the probability distributions in which we are chiefly interested have a dimension that is greater than the Cauchy dimension of 1, but less than the normal dimension of 2. (What is meant here by the "Cauchy dimension of 1" and the "normal dimension of 2" will be clarified as we go along.)

Notes

[1] See Chapter 14 in William Feller, *An Introduction to Probability Theory and Its Applications*, Vol. I, 3rd ed., John Wiley & Sons, New York, 1968.

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from *The Laissez Faire City Times*, Vol 3, No 27, July 5, 1999

Coin Toss Plot

If you can read this, then your browser is not set up for Java. . .

The applet shows 3200 successive coin tosses, presented in 8 rows of 400 tosses each. The height of the image is the number of heads minus the number of tails (the "winnings" of Henry who wins a dollar if heads comes up, but losses a dollar if tails comes up.) Because of space constraints, the rows may overlap if the cumulative wins or losses depart far from zero. (The eight rows are spaced 50 pixels apart, so a win of 25 on one row will touch a loss of 25 on the previous row.) Often wins or losses become so great that solid blocks of blue are formed.

Hit "Reload" or "Refresh Screen" on your browser to do a new series of 3200 coin tosses. Each series of 3200 coin tosses starts from a value of zero.

Here is the java source code.

Coin Toss Zero Set Plot

If you can read this, then your browser is not set up for Java. . .

The applet shows the **zero set** of 3200 successive coin tosses, presented in 8 rows of 400 tosses each. The **zero set** is the set of values of **n** where the number of heads and the number of tails are equal. It corresponds to the times when Henry, a gambler, reaches **a *break-even* point**. Henry wins a dollar if heads comes up, but losses a dollar if tails comes up, so the zero set corresponds to those times when his wins and losses are equalized. Note that the zero set has a Cantor-dust-like structure.

Hit "Reload" or "Refresh Screen" on your browser to do a new series of 3200 coin tosses. Each series of 3200 coin tosses starts from a value of zero.

Here is the java source code.

Normal Minus Cauchy Probability Density

If you can read this, then your browser is not set up for Java. . .

This applet plots 400 points as **x** varies from x = -10 to x = +10. The vertical axis shows the value of **f(x) minus g(x)**, where f(x) is the normal density, and g(x) is the Cauchy density. The blue area corresponds to the region f(x)>g(x), while the red area corresponds to the region g(x)>f(x). The Cauchy distribution has much greater probability in the extreme tails.

Here is the java source code.

Chaos and Fractals in Financial Markets

Part 5

by J. Orlin Grabbe

Louis Bachelier Visits the New York Stock Exchange

Louis Bachelier, resurrected for the moment, recently visited the New York Stock Exchange at the end of May 1999. He was somewhat puzzled by all the hideous concrete barriers around the building at the corner of Broad and Wall Streets. For a moment he thought he was in Washington, D.C., on Pennsylvania Avenue.

Bachelier was accompanied by an angelic guide named Pete. "The concrete blocks are there because of Osama bin Ladin," Pete explained. "He's a terrorist." Pete didn't bother to mention the blocks had been there for years. He knew Bachelier wouldn't know the difference.

"Terrorist?"

"You know, a ruffian, a scoundrel."

"Oh," Bachelier mused. "Bin Ladin. The son of Ladin."

"Yes, and before that, there was Abu Nidal."

"Abu Nidal. The father of Nidal. Hey! Ladin is just Nidal spelled backwards. So we've gone from the father of Nidal to the son of backwards-Nidal?"

"Yes," Pete said cryptically. "The spooks are never too creative when they are manufacturing the boogeyman of the moment. If you want to understand all this, read about 'Goldstein' and the daily scheduled 'Two Minutes Hate' in George Orwell's book *1984*."

"1984? Let's see, that was fifteen years ago," Bachelier said. "A historical work?"

"Actually, it's futuristic. But he who controls the present controls the past, and he who controls the past controls the future."

Bachelier was mystified by the entire conversation, but once they got inside and he saw the trading floor, he felt right at home. Buying, selling, changing prices. The chalk boards were now electric, he saw, and that made the air much fresher.

"Look," Bachelier said, "the Dow Jones average is still around!"

"Yes," nodded Pete, "but there are a lot of others also. Like the S&P500 and the New York Stock Exchange Composite Index."

"I want some numbers!" Bachelier exclaimed enthusiastically. Before they left, they managed to con someone into giving them the closing prices for the NYSE index for the past 11 days.

"You can write a book," Pete said. "Call it *Eleven Days in May*. Apocalyptic titles are all the rage these days—except in the stock market."

Bachelier didn't pay him any mind. He had taken out a pencil and paper and was attempting to calculate logarithms through a series expansion. Pete watched in silence for a while, before he took pity and pulled out a pocket calculator.

"Let me show you a really neat invention," the angel said.

Bachelier's Scale for Stock Prices

Here is Bachelier's data for eleven days in May. We have the calendar date in the first column of the table; the NYSE Composite Average, S(t), in the second column; the log of S(t) in the third column; the change in log prices, x(t) = log S(t) – log S(t-1) in the fourth column; and $x(t)^2$ in the last column. The sum of the variables in the last column is given at the bottom of the table.

Date	S(t)	log S(t)	x(t)	$x(t)^2$
May 14	638.45	6.459043		
May 17	636.92	6.456644	-.002399	.000005755
May 18	634.19	6.452348	-.004296	.000018456

May 19	639.54	6.460749	.008401	.000070577
May 20	639.42	6.460561	-.000188	.000000035
May 21	636.87	6.456565	-.003996	.000015968
May 24	626.05	6.439430	-.017135	.000293608
May 25	617.34	6.425420	-.014010	.000196280
May 26	624.84	6.437495	.012075	.000145806
May 27	614.02	6.420027	-.017468	.000305131
May 28	622.26	6.433358	.013331	.000177716
sum of all $x(t)^2$ = .001229332				

What is the meaning of all this?

The variables x(t), which are the one-trading-day changes in log prices, are the variables in which Bachelier is interested for his theory of Brownian motion as applied to the stock market:

x(t) = log S(t) – log S(t-1).

Bachelier thinks these should have a normal distribution. Recall from Part 4 that a normal distribution has a location parameter **m** and a scale parameter **c**. So what Bachelier is trying to do is to figure out what **m** and **c** are, assuming that each day's **m** and **c** are the same as any other day's.

The location parameter **m** is easy. It is zero, or pretty close to zero.

In fact, it is not quite zero. Essentially there is a drift in the movement of the stock index **S(t)**, given by the difference between the interest rate (such as the broker-dealer loan rate) and the dividend yield on stocks in the average.[1] But this is tiny over our eleven trading days (which gives us ten values for **x(t)**). So Bachelier just assumes **m** is zero.

So what Bachelier is doing with the data is trying to estimate **c**.

Recall from Part 2 that if today's price is **P**, Bachelier modeled the *probability interval* around the log of the price change by

(log P – **a** $T^{0.5}$, log P + **a** $T^{0.5}$), for some constant **a**.

But now, we are writing our stock index price as **S**, not **P**; and the constant **a** is just our scale parameter **c**. So, changing notation, Bachelier is interested in the probability interval

(log **S** – **c** $T^{0.5}$, log **S** + **c** $T^{0.5}$), for a given scale parameter **c**.

One way of estimating the scale **c (c** is also called the "standard deviation" in the context of the normal distribution) is to add up all the squared values of **x(t)**, and take the average (by dividing by the number of observations). This gives us an estimate of the variance, or $\mathbf{c^2}$. Then we simply take the square root to get the scale **c** itself. (This is called a *maximum likelihood estimator* for the standard deviation.)

Adding up the terms in the right-hand column in the table gives us a value of .001229332. And there are 10 observations. So we have

variance = $\mathbf{c^2}$ = .001229332/10 = .0001229332.

Taking the square root of this, we have

standard deviation = **c** = $(.0001229332)^{0.5}$ = .0110875.

So Bachelier's changing probability interval for log **S** becomes:

(log **S** – **.0110875** $T^{0.5}$, log **S** + **.0110875** $T^{0.5}$).

To get the probability interval for the price **S** itself, we just take exponentials (raise to the power **exp** = e = 2.718281…), and get

(**S** exp(– .0110875 $T^{0.5}$), **S** exp(.0110875 $T^{0.5}$)).

Since the current price on May 28, from the table, is 622.26, this interval becomes:

(622.26 exp(– .0110875 $T^{0.5}$), 622.26 exp(.0110875 $T^{0.5}$)).

"This expression for the probability interval tells us the probability distribution over the next T days," Bachelier explained to Pete. "Now I understand what you meant. He who controls the present controls the past, because he can obtain past data. While he who masters this past data controls the future, because he can calculate future probabilities!"

"Umm. That wasn’t what I meant," the angel replied. "But never mind."

Over the next 10 trading days, we have $T^{0.5} = 10^{0.5} = 3.162277$. So substituting that into the probability interval for price, we get

(622.26 (.965545), 622.26 (1.035683)) = (600.82, 644.46).

This probability interval gives a price range for plus or minus one scale parameter (in logs) **c**. For the normal distribution, that corresponds to **68 percent probability**. With 68 percent probability, the price will lie between 600.82 and 644.46 at the end of 10 more trading days, according to this calculation.

To get a **95 percent probability** interval, we use plus or minus **2c,**

(622.26 exp(– (2) .0110875 $T^{0.5}$), 622.26 exp((2) .0110875 $T^{0.5}$)),

which gives us a price interval over 10 trading days of

(580.12, 667.46).

Volatility

In the financial markets, the scale parameter **c** is often called "volatility". Since a normal distribution is usually assumed, "volatility" refers to the standard deviation.

Here we have measured the scale **c**, or volatility, on a basis of one trading day. The value of **c** we calculated, c = .0110875, was calculated over 10 trading days, so it would be called in the markets "a 10-day historical volatility." If calculated over 30 past trading days, it would be "a 30-day historical volatility."

However, market custom would dictate two criteria by which volatility is quoted:

1. quote volatility at an *annual* (not daily) rate;
2. quote volatility in *percentage* (not decimal) terms.

To change our daily volatility **c** = .0110875 into annual terms, we note that there are about 256 trading days in the year. The square root of 256 is 16, so to change daily volatility into annual volatility, we simply multiply it by 16:

annual **c** = 16 (daily **c**) = 16 (.0110875) = .1774.

Then we convert this to percent (by multiplying by 100 and calling the result "percent"):

annual **c** = 17.74 percent.

The New York Stock Exchange Composite Index had a historical volatility of 17.74 percent over the sample period during May.

Note that an *annual* volatility of 16 percent corresponds to a *daily* volatility of 1 percent. This is a useful relationship to remember, because we can look at a price or index, mentally divide by 100, and say the price change will fall in the range of plus or minus that amount with 2/3 probability (approximately). For example, if the current gold volatility is 16 percent, and the price is $260, we can say the coming day’s price change will fall in the range of plus or minus $2.60 with about 2/3 probability.

Notice that 256 trading days give us a probability interval that is only 16 times as large as the probability interval for 1 day. This translates into a Hausdorff dimension for time (in the probability calculation) as D = log(16)/log(256) = ½ or 0.5, which is just the Bachelier-Einstein square-root-of-T ($T^{0.5}$) law.

The way we calculated the scale **c** is called "historical volatility," because we used actual historical data to estimate **c**. In the options markets, there is another measure of volatility, called "**implied volatility**." Implied volatility is found by back-solving an option value (using a valuation formula) for the volatility, **c**, that gives the current option price. Hence this volatility, which pertains to the future (specifically, to the future life of the option) is *implied* by the price at which the option is trading.

Fractal Sums of Random Variables

Now for the fun part. We have been looking at random variables **x(t)** (representing changes in the log of price).

Under the assumption these random variables were normal, we estimated a scale parameter **c**, which allows us to do probability calculations.

In order to estimate **c**, we took the sums of random variables (or, in this instance, the sums of squares of x(t)).

Were our calculations proper and valid? Do they make any sense? The answer to these questions depends on the issue of *the probability distribution of a sum of random variables*. How does the distribution of the **sum** relate to the distributions of the individual random variables that are added together?

In answering this question we want to focus on ways we can come up with a location parameter **m**, and a scale parameter **c**. For the normal distribution, **m** is the *mean*, but for the Cauchy distribution the mean doesn't exist ("is infinite"). For the normal distribution, the scale parameter **c** is the standard deviation, but for the Cauchy distribution the standard deviation doesn't exist. Nevertheless, a location **m** and a scale **c** exist for the Cauchy distribution. The maximum likelihood estimator for **c** will not be the same in the case of the Cauchy distribution as it was for the normal. We can't take squares if the x(t) have a Cauchy distribution.

Suppose we have **n** random variables $\mathbf{X}_i$, *all with the same distribution*, and we calculate their sum **X**:

$$\mathbf{X} = X_1 + X_2 + \ldots + X_{n-1} + X_n.$$

Does the distribution of the sum **X** have a simple form? In particular, can we relate the distribution of **X** to the common distribution of the $\mathbf{X_i}$? Let's be even more specific. We have looked at the normal (Gaussian) and Cauchy distributions, both of which were parameterized with a location **m** and scale **c**. If each of the $\mathbf{X_i}$ has a location **m** and scale **c**, whether normal or Cauchy, can that information be translated into a location and a scale for the sum **X**?

The answer to all these questions is *yes*, for a class of distributions called *stable distributions*. (They are also sometimes called "Levy stable", "Pareto-Levy", or "stable Paretian" distributions.) Both the *normal* and the *Cauchy* are stable distributions. But there are many more.

We will use the notation "~" as shorthand for "has the same distribution as." For example,

$X_1 \sim X_2$

means X_1 and X_2 have the same distribution. We now use "~" in the following definition of stable distributions:

Definition: A random variable X is said to have a **stable distribution** if for any **n** >= 2 (greater than or equal to 2), there is a positive number C_n and a real number D_n such that

$X_1 + X_2 + \ldots + X_{n-1} + X_n \sim \mathbf{C_n} X + \mathbf{D_n}$

where $X_1, X_2, \ldots, X_n$ are all independent copies of X.

Think of what this definition means. If their distribution is stable, then the sum of **n** identically distributed random variables has the same distribution as any one of them, except by multiplication by a scale factor C_n and a further adjustment by a location D_n .

Does this remind you of fractals? Fractals are geometrical objects that look the same at different scales. Here we have random variables whose probability distributions look the same at different scales (except for the add factor D_n).

Let’s define two more terms.[2]

Definition: A stable random variable X is **strictly stable** if $D_n = 0$.

So strictly stable distributions are clearly* fractal *in nature, because the sum of n independent copies of the underlying distribution looks exactly the same as the

underlying distribution itself, once adjust by the scale factor C_n. One type of strictly stable distributions are *symmetric* stable distributions.

Definition: A stable random variable X is **symmetric stable** if its distribution is symmetric—that is, if **X** and -**X** have the same distribution.

The scale parameter C_n necessarily has the form [3]:

$$\mathbf{C_n = n^{1/\alpha}, \text{ where } 0 < \alpha <= 2.}$$

So if we have **n** independent copies of a symmetric stable distribution, their sum has the same distribution with a scale that is $\mathbf{n}^{1/\alpha}$ times as large.

For the normal or Gaussian distribution, $\alpha = 2$. So for **n** independent copies of a normal distribution, their sum has a scale that is $\mathbf{n}^{1/\alpha} = \mathbf{n}^{1/2}$ times as large.

For the Cauchy distribution, $\alpha = 1$. So for **n** independent copies of a Cauchy distribution, their sum has a scale that is $\mathbf{n}^{1/\alpha} = \mathbf{n}^{1/1} = \mathbf{n}$ times as large.

Thus if, for example, Brownian particles had a Cauchy distribution, they would scale not according to a $\mathbf{T}^{0.5}$ law, but rather according to a **T** law!

Notice that we can also calculate a Hausdorff dimension for symmetric stable distributions. If we divide a symmetric stable random variable **X** by a scale factor of $\mathbf{c} = \mathbf{n}^{1/\alpha}$, we get the probability equivalent [4] of N = **n** copies of $\mathbf{X/n}^{1/\alpha}$. So the Hausdorff dimension is

$$D = \log N / \log c = \log \mathbf{n} / \log(\mathbf{n}^{1/\alpha}) = \alpha .$$

This gives us a simple interpretation of α. The parameter α is simply the Hausdorff dimension of a symmetric stable distribution. For the normal, the Hausdorff dimension is equal to 2, equivalent to that of a plane. For the Cauchy, the Hausdorff dimension is equal to 1, equivalent to that of a line. In between is a full range of values, including symmetric stable distributions with Hausdorff dimensions equivalent to the Koch Curve (log 4/log 3) and the Sierpinski Carpet (log 8/log3).

Some Fun with Logistic Art

Now that we've worked our way to the heart of the matter, let's take a break from probability theory and turn our attention again to dynamical systems. In particular, let's look at our old friend the logistic equation:

$x(n+1) = \mathbf{k}\ x(n)\ [1 - x(n)]$,

where x(n) is the input variable, x(n+1) is the output variable, and **k** is a constant.

In Part 1, we looked at a particular version of this equation where k = 4. In general, **k** takes values $0 < k <= 4$.

The dynamic behavior of this equation depends on the value **k**, and also on the particular starting value or starting point, **x(0)**. Later in this series we will examine how the behavior of this equation changes as we change **k**. But not now.

Instead, we are going to look at this equation when we substitute for **x**, which is a real variable, a complex variable **z**:

$z(n+1) = k\ z(n)\ [1 - z(n)]$.

Complex numbers z have the form

$z = x + \mathbf{i}\ y$,

where **i** is the square root of minus one. Complex numbers are normally graphed in a plane, with **x** on the horizontal ("real") axis, while **y** is on the vertical ("imaginary") axis.

That means when we iterate **z**, we actually iterate *two* values: **x** in the horizontal direction, and **y** in the vertical direction. The complex logistic equation is:

$x + i\ y = k\ (x + i\ y)\ [\ 1 - (x + i\ y)]$.

(Note that I have dropped the notation x(n) and y(n) and just used **x** and **y**, to make the equations easier to read. But keep in mind that **x** and **y** on the left-hand side of the equation represent *output*, while the **x** and **y** on the right-hand side of the equation represent *input*.)

The output **x**, the *real* part of **z**, is composed of all the terms that do not multiply **i**, while the output **y**, the *imaginary* part of **z**, is made up of all the terms that multiply **i**.

To complete the transformation of the logistic equation, we let **k** be complex also, and write

k = A + B i,

giving as our final form:

x + i y = (A + B i) (x + i y) [1 – (x + i y)].

Now we multiply this all out and collect terms. The result is two equations in x and y:

$x = A (x-x^2+y^2) + B (2xy-y)$
$y = B (x-x^2+y^2) - A (2xy-y)$.

As in the real version of the logistic equation, the behavior of the equation depends on the multiplier k = A + B i (that is, on A and B), as well as the initial starting value of z = x + i y (that is, on x(0) and y(0)).

Julia Sets

Depending on **k**, some beginning values z(0) = x(0) + **i** y(0) blow off to infinity after a certain number of iterations. That is, the output values of **z** keep getting larger and larger, diverging to infinity. As **z** is composed of both an **x** term and a **y** term, we use as the criterion for "getting large" the value of

$x^2 + y^2$.

The square root of this number is called the *modulus* of **z**, and represents the length of a vector from the origin (0,0) to the point **z** = (x,y). In the iterations we are about to see, the criterion to determine if the equation is diverging to infinity is

$x^2 + y^2 > 4$,

which implies the modulus of **z** is greater than 2.

When the equation is iterated, some starting values diverge to infinity and some don’t. **The *Julia set* is the set of starting values for z that remain finite under iteration**. That is, the Julia set is the set of all starting values (x(0), y(0)) such that the equation output does not blow off to infinity as the equation is iterated.

Each value for **k** will produce a different Julia set (i.e., a different set of (x(0) ,y(0)) values that do not diverge under iteration).

Let's do an example. Let **k** = 1.678 + .95 **i**. That is, A = 1.678 and B = .95. We let starting values for x(0) range from –.5 to 1.5, while letting starting values for y(0) range from -.7 to +.7.

We keep **k** constant always, so we are graphing the Julia set associated with **k** = 1.678 + .95 **i**.

We iterate the equation 256 times. If, at the end of 256 iterations, the modulus of **z** is not greater than 2, we paint the starting point (x(0), y(0)) black. So the entire **Julia set in this example is colored black**. If the modulus of **z** exceeds 2 during the iterations, the starting point (x(0), y(0)) is ***assigned a color depending on the rate the equation is blowing off to infinity***.

To see the demonstration, be sure Java is enabled on your web browser and click here.

We can create a plot that looks entirely different by making a different color assignment. For the next demonstration, we again iterate the dynamical system 256 times for different starting values of **z(n)**. If, during the iterations, the modulus of **z** exceeds 2, then we know the iterations are diverging, so we plot the starting value **z(0) = (x(0), y(0))** black. Hence the **black region of the plot is made up of all the points *not* in the Julia set**. For the Julia set itself, we assign bright colors. The color assigned depends on the value of **z** after 256 iterations. For example, if the square of the modulus of **z** is greater than .6, but less than .7, the point z(0) is assigned a light red color. Hence the **colors in the Julia set indicate the value of the modulus of z at the end of 256 iterations.**

To see the second demonstration of the same equation, but with this alternative color mapping, be sure Java is enabled on your web browser and click here

So, from the complex logistic equation, a dynamical system, we have created a fractal. The border of the Julia set is determined by **k** in the equation, and this border was created in a working Euclidean space of 2 dimensions, has a topological dimension of 1, but has a Hausdorff dimension that lies between these two numbers.

Meanwhile, we have passed from mathematics to art. Or maybe the art was there all along. We just had to learn how to appreciate it.

Notes

[1] This is the stock market equivalent of the Interest Parity Theorem that relates the forward price F(t+T) of a currency, T-days in the future, to the current spot price S(t). In the foreign exchange market, the relationship can be written as:

$$F(t+T) = S(t)\,[1 + r\,(T/360)]/[1+r^*(T/360)]$$

where r is the domestic interest rate (say the dollar interest rate), and r* is the foreign interest rate (say the interest rate on the euro). S is then the spot dollar price of the euro, and F is the forward dollar price of the euro.

We can also use this equation to give us the forward value F of a stock index in relation to its current value S, in which case r* must be the dividend yield on the stock index.

(A more precise calculation would disaggregate the "dividend yield" into the actual days and amounts of dividend payments.)

This relationship is explored at length in Chapter 5, "Forwards, Swaps, and Interest Parity," in J. Orlin Grabbe, *International Financial Markets*, 3rd edition, Prentice-Hall, 1996.

[2] The definitions here follow those in Gennady Samorodnitsky and Murad S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman & Hall, New York, 1994.

[3] This is Theorem VI.1.1 in William Feller, *An Introduction to Probability Theory and Its Applications*, Vol 2, 2nd ed., Wiley, New York, 1971.

[4] If Y = $\mathbf{X/n^{1/\alpha}}$, then for **n** independent copies of Y,

$Y_1 + Y_2 + \ldots + Y_{n-1} + Y_n \sim \mathbf{n^{1/\alpha}\,Y = n^{1/\alpha}\,(X/n^{1/\alpha})} = \mathbf{X}$.

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from *The Laissez Faire City Times*, Vol 3, No 29, July 19, 1999

Julia Plot of the Complex Logistic Equation

If you can read this, then your browser is not set up for Java. . .

This applet plots the Julia set for the complex logistic equation: z(n+1) = k z(n)[1-z(n)], for k = 1.678 + .95 **i**. The equation is iterated 256 times for different starting values of z(n). If the equation is not diverging to infinity at the end of the 256 iterations, then the starting point z(0) is painted black. Hence the **black area of the plot represents the Julia set** (except for some points on the outer corners that are also painted black if they diverge on the first iteration). If, however, the equation begins diverging (defined as occurring when the modulus of z(n+1) > 2), the iterations are terminated, and the point is plotted with one of 16 colors. The iterations cycle through this set of colors, but terminate when the equation diverges. Hence, **the color plotted indicates the rate the equation output is blowing off to infinity**.

The Java source code.

Alternative Julia Plot of the Complex Logistic Equation

If you can read this, then your browser is not set up for Java. . .

This applet plots the Julia set for the complex logistic equation: z(n+1) = k z(n)[1-z(n)], for k = 1.678 + .95 **i**. The equation is iterated 256 times for different starting values of z(n). If the equation is not diverging to infinity at the end of the 256 iterations, then the terminal value z(n+1) = z(256) is assigned a color based on the value of the square of its modulus (i.e., on the value of x^2+y^2). Hence, **the brightly colored area of the plot consists of the Julia set**. If, however, the equation begins diverging (defined as occurring when the modulus of z(n+1) > 2), the iterations are terminated, and the point z(0) is painted black. Hence **the black area of the plot represents points that diverge and which are therefore not in the Julia set.**

The Java source code.

Chaos and Fractals in Financial Markets

Part 6

by J. Orlin Grabbe

Prechter's Drum Roll

Robert Prechter is a drummer. He faces the following problem. He wants to strike his drum three times, creating two time intervals which have a special ratio:

1<------------***g***-------------->2<--------------------***h***-------------------->3

Here is the time ratio he is looking for: he wants the ratio of the first time interval to the second time interval to be the same as the ratio of the second time interval to the entire time required for the three strikes.

Let the first time internal (between strikes 1 and 2) be labeled g, while the second time interval (between strikes 2 and 3) be labeled h. So what Prechter is looking for is the ratio of g to h to be the same as h to the whole. However, the whole is simply $g + h$, so Prechter seeks g and h such that:

$$g / h = h / (g+h).$$

Now. Prechter is only looking for a particular ratio. He doesn't care whether he plays his drum slow or fast. So h can be anything: 1 nano-second, 1 second, 1 minute, or whatever. So let's set $h = 1$. (Note that by setting $h = 1$, we are choosing our **unit of measurement**.) We then have

$$g / 1 = 1 / (1+g).$$

Multiplying the equation out we get

$$g^2 + g - 1 = 0.$$

This gives two solutions:

$g = [- 1 + 5^{0.5}] / 2 = 0.618033…$, and

$g = [- 1 - 5^{0.5}] / 2 = -1.618033…$

The first, positive solution **($g = 0.618033…$)** is called the **golden mean**. Using $h = 1$ as our scale of measurement, then g**,** the golden mean**,** is the solution to the ratio

$\boldsymbol{g / h} = \boldsymbol{h / (g+h)}$.

By contrast, if we use ***g*** = 1 as our scale of measurement, and solve for ***h***, we have

$\mathbf{1} / \boldsymbol{h} = \boldsymbol{h} / (\mathbf{1}+\boldsymbol{h})$, which gives the equation

$\boldsymbol{h}^2 - \boldsymbol{h} - \mathbf{1} = \mathbf{0}.$

Which gives the two solutions:

$\boldsymbol{h} = [\ 1 + 5^{0.5}] / 2 = 1.618033\ldots$, and

$\boldsymbol{h} = [\ 1 - 5^{0.5}] / 2 = -0.618033\ldots$

Note that since the units of measurement are somewhat aribitrary, ***h*** has as much claim as ***g*** to being the solution to Prechter's drum roll. Naturally, ***g*** and ***h*** are closely related:

h (using ***g*** as the unit scale) = 1/ ***g*** **(**using ***h*** as the unit scale).

for either the positive or negative solutions:

1.618033… = 1/ 0.618033…

-0.618033… = 1/ -1.618033.

What is the meaning of the negative solutions? These also have a physical meaning, depending on where we place our ***time origin***. For example, let's let the second strike of the drum be time **t**=0:

<------------***g***------------->0<------------------***h***------------------->

Then we find that for ***g*** = -1.618033, ***h*** = 1, we have

-1.618033 /1 = 1/ [1 - 1.618033].

So the negative solutions tell us the same thing as the positive solutions; but they correspond to a time origin of **t** = 0 for the second strike of the drum.

The same applies for ***g*** = 1, ***h*** = -0.618033, since

1 / -0.618033 = -0.618033/(1 – 0.618033),

but in this case **time is running backwards**, not forwards.

The golden mean ***g***, or its reciprocal equivalent ***h***, are found throughout the natural world. Numerous books have been devoted to the subject. These same ratios are found in financial markets.

Symmetric Stable Distributions and the Golden Mean Law

In Part 5, we saw that symmetric stable distributions are a type of probability distribution that are fractal in nature: a sum of **n** independent copies of a symmetric stable distribution is related to each copy by a scale factor $\mathbf{n^{1/\alpha}}$**,** where α is the Hausdorff dimension of the given symmetric stable distribution.

In the case of the normal or Gaussian distribution, the Hausdorff dimension α **= 2**, which is equivalent to the dimension of a plane. A Bachelier process, or Brownian motion (as first covered in Part 2), is governed by a $T^{1/\alpha} = T^{1/2}$ law.

In the case of the Cauchy distribution (Part 4), the Hausdorff dimension α **= 1**, which is equivalent to the dimension of a line. A Cauchy process would be governed by a $T^{1/\alpha} = T^{1/1} = T$ law.

In general, $0 < \alpha <= 2$. This means that *between* the Cauchy and the Normal are all sorts of interesting distributions, including ones having the same Hausdorf dimension as a Sierpinski carpet (α = log 8/ log 3 = 1.8927….) or Koch curve (α = log 4/ log 3 = 1.2618….).

Interestingly, however, many financial variables are symmetric stable distributions with an α parameter that hovers around the value of ***h*** = 1.618033, where ***h*** is the reciprocal of the golden mean ***g*** derived and discussed in the previous section. This implies that these market variables follow a time scale law of $T^{1/\alpha} = T^{1/h} = T^{g} = T^{\mathbf{0.618033..}}$. **That is, these variables following a T-to-the-golden-mean power law, by contrast to Brownian motion, which follows a T-to-the-one-half power law.**

For example, I estimated α for daily changes in the dollar/deutschemark exchange rate for the first six years following the breakdown of the Bretton Woods Agreement of fixed exchange rates in 1973. [1] (The time period was July 1973 to June 1979.) The value of α was calculated using maximum likelihood techniques [2]. The value I found was

α = 1.62

with a margin of error of plus or minus .04. You can’t get much closer than that to α **=** ***h*** **=** 1.618033…

In this and other financial asset markets, it would seem that time scales not according to the commonly assumed square-root-of-T law, but rather to a T^{g} law.

The Fibonacci Dynamical System

The value of ***h*** **=** 1.618033… is closely related to the ***Fibonacci*** sequence of numbers. The Fibonacci sequence of numbers is a sequence in which each number is the sum of the previous two:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

Notice the third number in the sequence, 2=1+1. The next number 3=2+1. The next number 5=3+2. And so on, each number being the sum of the two previous numbers.

This mathematical sequence appeared 1202 A.D. in the book *Liber Abaci*, written by the Italian mathematician Leonardo da Pisa, who was popularly known as Fibonacci (son of Bonacci). Fibonacci told a story about rabbits. These were *mathematical rabbits* that live forever, take one generation to mature, and always thereafter have one off-spring per generation. So if we start with 1 rabbit (the first 1 in the Fibonaaci sequence), the rabbit takes one generation to mature (so there is still 1 rabbit the next generation—the second 1 in the sequence), then it has a baby rabbit in the following generation (giving 2 rabbits—the 2 in the sequence), has another offspring the next generation (giving 3 rabbits); then, in the next generation, the first baby rabbit has matured and also has a baby rabbit, so there are two offspring (giving 5 rabbits in the sequence), and so on.

Now, the Fibonacci sequence represents the path of a dynamical system. We introduced dynamical systems in Part 1 of this series. (In Part 5, we discussed the concept of Julia Sets, and used a particular dynamical system—the complex logistic equation—to create computer art in real time using Java applets. The Java source code was also included.)

The Fibonacci dynamical system look like this:

F(n+2) = F(n+1) + F(n).

The number of rabbits in each generation (F(n+2)) is equal to the sum of the rabbits in the previous two generations (represented by F(n+1) and F(n)). This is an example of a more general dynamical system that may be written as:

F(n+2) = p F(n+1) + q F(n),

where p and q are some numbers (parameters). The solution to the system depends on the values of p and q, as well as the starting values F(0) and F(1). For the Fibonacci system, we have the simplification p = q = F(0) = F(1) = 1.

I will not go through the details here, but the Fibonacci system can be solved to yield the solution:

$F(n) = [1/5^{0.5}] \{ [(1+5^{0.5})/2]^n – [(1-5^{0.5})/2]^n \}, n = 1, 2, . . .$

The following table gives the value of F(n) for the first few values of n:

n	1	2	3	4	5
F(n)	1	1	2	3	5

And so on for the rest of the numbers in the Fibonacci sequence. Notice that the general solution involves the two solution values we previously calculated for ***h.*** To simplify, however, we will now write everything in terms of the first of these values (namely, ***h*** = 1.618033 …). Thus we have

$\boldsymbol{h} = [1 + 5^{0.5}] / 2 = 1.618033…$, and

$\boldsymbol{- 1/ h} = [1 - 5^{0.5}] / 2 = -0.618033…$

Inserting these into the solution for the Fibonacci system F(n), we get

$F(n) = [1/5^{0.5}] \{ [\boldsymbol{h}]^n – [\boldsymbol{-1/ h}]^n \}, n = 1, 2, . . .$

Alternatively, writing the solution using the golden mean ***g***, we have

$F(n) = [1/5^{0.5}] \{ [\boldsymbol{g}]^{-n} – [-\boldsymbol{g}]^n \}, n = 1, 2, . . .$

The use of Fibonacci relationships in financial markets has been popularized by Robert Prechter [3] and his colleagues, following the work of R. N. Elliott [4]. The empirical evidence that the Hausdorff dimension of some symmetric stable distributions encountered in financial markets is approximately $\alpha = \boldsymbol{h} =$ 1.618033… indicates that this approach is based on a solid empirical foundation.

Notes

[1] See "Research Strategy in Empirical Work with Exchange Rate Distributions," in J. Orlin Grabbe, *Three Essays in International Finance*, Ph.D. Thesis, Department of Economics, Harvard University, 1981.

[2] There are two key papers by DuMouchel which yield the

background needed for doing maximum likelihood estimates of α, where α < **2:**

DuMouchel, William H. (1973), "On the Asymptotic Normality of the Maximum Likelihood Estimate when Sampling from a Stable Distribution," *Annals of Statistics***,** 1, 948-57.

DuMouchel, William H. (1975), "Stable Distributions in Statistical Inference: 2. Information from Stably Distributed Samples," *Journal of the American Statistical Association*, 70, 386-393.

[3] See, for example:

Robert R. Prechter, Jr., *At the Crest of the Tidal Wave*, John Wiley & Sons, New York, 1995

Robert R. Prechter, Jr., *The Wave Principle of Human Social Behavior and the New Science of Socionomics*, New Classics Library, Gainesville, Georgia, 1999.

[4] See *R.N. Elliott's Masterworks—The Definitive Collection*, edited by Robert R. Prechter, Jr., Gainesville, Georgia, 1994.

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from *The Laissez Faire City Times*, Vol 3, No 35, September 6, 1999

Chaos and Fractals in Financial Markets

Part 7

by J. Orlin Grabbe

Grow Brain

Many dynamical systems create solution paths, or trajectories, that look strange and complex. These solution plots are called "strange attractors".

Some strange attractors have a fractal structure. For example, we saw in Part 3 that it was easy to create a fractal called a Sierpinski Carpet by using a *stochastic* dynamical system (one in which the outcome at each step is partially determined by a random component that either selects among equations or forms part of at least one of the equations, or both).

Here is a dynamical system that I ran across while doing computer art. I labeled it "Grow Brain" because of its structure. To see Grow Brain in action, make sure Java is enabled on your browser (you can turn it off afterward) and click here. (The truly paranoid can, of course, compile their own applet, since I provide the source code, as usual.)

The trajectory of Grow Brain is amazingly complex. But is it a fractal? That is, at some larger or smaller scale, will similar structures repeat themselves? Unlike the case of the Sierpinski Carpet, the answer to this question is not obvious for Grow Brain.

Some dynamical systems create fractal structures in *time* (as Brownian motion does, in Part 2, or the Fibonacci-type systems of Part 6 do), while others create fractal structures in *space* (as in the aforementioned Sierpinski carpet).

And some systems are all wet. Or maybe not, as the case may be.

Hurst, Hydrology, and the Annual Flooding of the Nile

For centuries, perhaps millennia, the yearly flooding of the Nile was the basis of agriculture which supported much of known civilization. The annual overflowing of the river deposited rich top soil from the Ethiopian Highland along the river banks. The water and silt were distributed by irrigation, and the staple crops of wheat, barley, and flax were planted. The grain was harvested and stored in silos and granaries, where it was protected from rodents by guard cats, whom the Egyptians worshipped and turned into a cult (of the goddess Bast) because of their importance for survival of the grain, and hence for human survival.

The amount of Nile flooding was critical. A good flood meant a good harvest, while a low-water flood meant a poor harvest and possible food shortage. The flooding came (and still comes) from tropical rains in the Upper Nile Basin in Ethiopia (the Blue Nile) and in the East African Plateau (the White Nile). The river flooding would begin in the Sudan in April, and reach Aswan in Egypt by July. (This would occur about the time of the heliacal rising of the Dog-Star Sirus, or Sothis, around July 19 in the Julian calendar.) The waters would then continue to rise, peaking in mid-September in Aswan. Further down the river at Cairo, the peak wouldn't occur until October. The waters would then fall rapidly in November and December, and continue to fall afterward, reaching their low point in the March to May period. Ancient Egypt had three seasons, all determined in reference to the river: *akhet*, the "inundation"; *peret*, the season when land emerged from the flood; and *shomu*, the time when water was low.

A British government bureaucrat named Hurst made a study of records of the Nile's flooding and noticed something interesting. Harold Edwin Hurst was a poor Leicester boy who made good, eventually working his way into Oxford, and later became a British "civil servant" in Cairo in 1906. He got interested in the Nile. He looked at 800 years of records and noticed that there was a tendency for a good flood year to be followed by another good flood year, and for a bad (low) flood year to be followed by another bad flood year.

That is, there appeared to be non-random runs of good or bad years. Later Mandelbrot and Wallis [1] used the term *Joseph effect* to refer to any persistent phenomenon like this (alluding to the seven years of Egyptian plenty followed by the seven years of Egyptian famine in the biblical story of Joseph).

Of course, even if the yearly flows were independent, there still could be runs of good or bad years. So to pin this down, Hurst calculated a variable which is now called a *Hurst exponent* **H**. The expectation was that **H** = ½ if the yearly flood levels were independent of each other.

Calculating the Hurst Exponent

Let me give a specific example of Hurst exponent calculation which will illustrate the general rule. Suppose there are 99 yearly observations of the height **h** of the mid-September Nile water level at Aswan: **h**(1), **h**(2), . . ., **h**(99).

Calculate a location **m** and a scale **c** for **h**. If we assume in general that **h** has a finite variance, then **m** is simply the sample mean of the 99 observations, while **c** is the standard deviation.

The first thing is to remove any trend, any tendency over the century for **h** to rise or fall as a long-run phenomena. So we subtract **m** from each of the observations **h**, getting a new series **x** that has mean zero:

$$\mathbf{x}(1) = \mathbf{h}(1) - \mathbf{m},$$
$$\mathbf{x}(2) = \mathbf{h}(2) - \mathbf{m},$$
$$\ldots$$
$$\mathbf{x}(99) = \mathbf{h}(99) - \mathbf{m} .$$

The set of **x**’s are a set of variables with mean zero. Positive **x**’s represent those years when the river level is above average, while negative **x**’s represent those years when the river level is below average.

Next we form partial sums of these random variables, each partial sum **Y**(n) being the sum of all the years prior to year **n**:

$$\mathbf{Y}(1) = x(1),$$
$$\mathbf{Y}(2) = x(1) + x(2),$$
$$. . .$$
$$\mathbf{Y}(n) = x(1) + x(2) + . . . + x(n),$$
$$. . .$$
$$\mathbf{Y}(99) = x(1) + x(2) + x(3) + . . . + x(99).$$

Since the **Y**'s are a sum of mean-zero random variables **x**, they will be positive if they have a preponderance of positive **x**'s and negative if they have a preponderance of negative **x**'s. In general, the set of **Y**'s

{**Y**(1), **Y**(2), …. , **Y**(99)}

will have a maximum and a minimum: **max Y** and **min Y**, respectively. The difference between these two is called the *range* **R**:

R = max Y - min Y

If we adjust **R** by the scale parameter **c**, we get the *rescaled range*:

rescaled range = **R/c** .

Now, the probability theorist William Feller [2] had proven that if a series of random variables like the **x**'s 1) had finite variance, and 2) were independent, then the rescaled range formed over **n** observations would be equal to:

$\mathbf{R/c} = k\ \mathbf{n}^{1/2}$

where k is a constant (in particular, $k = (\pi /2)^{1/2}$) . That is, the rescaled range would increase much like the partial sums of independent variables (with finite variance) we looked at in Part 5— namely, the partial sums would increase by a factor of $\mathbf{n}^{1/2}$.

In particular, for **n** = 99 in our hypothetical data, the result would be:

$\mathbf{R/c} = k\ 99^{1/2}$.

Now, the latter equation implies log(R/c) = log k + ½ log 99. So if you ran a regression of log(R/c) against log(n) [for a number of rescaled ranges (R/c) and their associated number of years n] so as to estimate an intercept **a** and a slope **b**,

log(R/c) = **a** + **b** log(n),

you should find that **b** is statistically indistinguishable from ½.

But that wasn't what Hurst found. Instead, he found that in general the rescaled range was governed by a power law

$\mathbf{R/c} = k\,\mathbf{n}^{\mathbf{H}}$

where the Hurst exponent **H** was *greater than* ½ (Hurst found H ≅ .7)

This implied that succeeding **x**'s were not independent of each other: **x**(t) had some sticky, persistent effect on **x**(t+1). This was what Hurst had observed in the data, and his calculation showed H to be a good bit above ½ [3].

That this would be true in general for H > ½ , of course, needs to be proven. Nevertheless, to summarize, for reference, for the Hurst exponent H:

> H = ½ : the flood level deviations from the mean are independent, random; the **x**'s are independent and correspond to a random walk;
>
> ½ < H <=1: the flood level deviations are *persistent*— high flood levels tend to be followed by high flood levels, and low flood levels by low flood levels; x(t+1) tends to deviate from the mean the same way x(t) did; the probability that x(t+1) deviates from the mean in the same direction as x(t) increases as H approaches 1;
>
> 0<=H< 1/2: the flood level deviations are *anti-persistent*— the **x**'s are mean-reverting; high flood levels have a tendency to be followed by low flood levels, and vice-versa; the probability that x(t+1) deviates from the mean in the opposite direction from x(t) increases as H approaches 0.

A Misunderstanding to Avoid

Recall that Bachelier had noted that the probability range of the log of a stock price would increase with the square root of time T. The probability range, starting at log S, would grow with T according to:

$(\log S - k\,T^{1/2}, \log S + k\,T^{1/2})$, where **k** is the scale (in his case, the standard deviation) **c**, **k** = **c**. But, more generally, the symmetric stable distributions of Part 5, increase with T raised to the reciprocal power of the Hausdorff dimension α (α <=2):

$$(\log S - k\, T^{1/\alpha}, \log S + k\, T^{1/\alpha}).$$

Hurst similarly said the rescaled range of the flood level varied according to (setting n = T):

$$R/c = k\, T^{\mathbf{H}}.$$

So it is tempting to equate the Hurst exponent H with the reciprocal of the Hausdorff dimension D, to equate H with $1/D = 1/\alpha$. But we must be careful.

Recall that symmetric stable distributions, with $\alpha < 2$, have infinite variance (for them, variance is a blob measure that is not meaningful). However, here in discussing the Hurst exponent we are assuming that the variance, and standard deviation (the scale **c**), are finite, and hence $\alpha = 2$. The role of the Hurst exponent is to inform us whether the yearly flood deviations are *independent* or *persistent*. H is *not* related to the need for a different scale measure. The variance and the standard deviation are well defined for these latter processes.

Nevertheless, the formal equation H = 1/D or D = 1/H yields the correct exponent for T in the case ½<= H <=1. Even though $\alpha = 2$, the calculation of the Hausdorf dimension D yields D<2 if the increments are not independent. Hence D can take a minimum value of 1, D = 1/H = 1/1 = 1 when H=1, so that the process accumulates variation (rescaled range) much like a Cauchy sequence ($T^{\mathbf{H}} = T$); or a maximum of 2, D = 1/H = 1/½ = 2 when H=½, so that the process accumulates variation (rescaled range) like a Gaussian sequence ($T^{\mathbf{H}} = T^{1/2}$), or ordinary Brownian motion. [4]

Mandelbrot called these types of processes where $\alpha = 2$, but where H ≠ ½, *fractional Brownian motion*. (I will not here elaborate the case H < ½.)

Bull and Bear Markets

We are, of course, used to the idea of persistent phenomena in the stock market and foreign exchange markets. The NASD rises relentlessly for a period of time. Then it falls just as persistently. There are bull and bear markets, implying the price rise or decline is a *persistent* phenomena, and not just an accidental accumulation of random variables in one direction.

The US dollar rises relentless for a period of years, then (as it is doing now) begins a relentless decline for another period of years. In the case of the Nile, the patterns of rising and falling are partly governed by the weather patterns in the green rain forest of the Ethiopian highlands. In the case of the US dollar, the patterns of rising and falling are partly governed by the span of Green in the Washington D.C. lowlands.

Notes

[1] B.B. Mandelbrot & J. R. Wallis, "Noah, Joseph, and Operational Hydrology." *Water Resources Research* **4**, 909-918, (1968).

[2] W. Feller, "The asymptotic distribution of the range of sums of independent random variables." *Annals of Mathematical Statistics* **22**, 427 (1951).

[3] H. E. Hurst, "Long-term storage capacity of reservoirs." *Tr. of the American Society of Civil Engineers* **116**, 770-808 (1951).

[4] See also the discussion on pages 251-2 in Benoit B. Mandelbrot, *The Fractal Geometry of Nature*. W.H. Freeman and Company, New York, 1983.

J. Orlin Grabbe is the author of International Financial Markets, and is an internationally recognized derivatives expert. He has recently branched out into cryptology, banking security, and digital cash. His home page is located at http://orlingrabbe.org/ .

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from *The Laissez Faire City Times*, Vol 5, No 3, January 15, 2001

Iterative Plot of Grow Brain

If you can read this, then your browser is not set up for Java. . .

Wait a minute for the applet to run. This amazingly complex structure is created by the simple iteration of the two equations:

> X = Y - (X/|X|)*sqrt(|aX|)
> Y = b - X

(Here "sqrt" means to take the square root.)

The first series of iterations here starts with the initial values X = .9 and Y = 0. The two equations are then iterated 10,000 times, plotting each (X,Y) point of the iteration. Then the starting value for Y is increased slightly, and another 10,000 plots are plotted (see the Java source code for details).

By varying the parameter **a** in the first equation above (**a**X is set at 1.53 X for the applet supplied here), and also the rate at which the starting value for Y is increased (set Y = .025*col, say, instead of Y = .047*col), different structures can be obtained.

The Java source code.

Chaos and Fractals in Financial Markets

Part 2

by J. Orlin Grabbe

The French Gambler and the Pollen Grains

In 1827 an English botanist, Robert Brown got his hands on some new technology: a microscope "made for me by Mr. Dolland, . . . of which the three lenses that I have generally used, are of a 40th, 60th, and 70th of an inch focus."

Right away, Brown noticed how pollen grains suspended in water jiggled around in a furious, but random, fashion.

To see what Brown saw under his microscope, make sure that Java is enabled on your web browser, and then click here.

What was going on was a puzzle. Many people wondered: Were these tiny bits of organic matter somehow ***alive***? Luckily, Hollywood wasn't around at the time, or John Carpenter might have made his wonderful horror film *They Live!* about pollen grains rather than about the infiltration of society by liberal control-freaks.

Robert Brown himself said he didn't think the movement had anything to do with tiny currents in the water, nor was it produced by evaporation. He explained his observations in the following terms:

> "That extremely minute particles of solid matter, whether obtained from organic or inorganic substances, when suspended in pure water, or in some other aqueous fluids, exhibit motions for which I am unable to account, and from which their irregularity and seeming independence resemble in a remarkable degree the less rapid motions of some of the simplest animalcules of infusions. That the smallest moving particles observed, and which I have termed Active Molecules, appear to be spherical, or nearly so, and to be between 1-20,0000dth and 1-30,000dth of an inch in diameter; and

> that other particles of considerably greater and various size, and either of similar or of very different figure, also present analogous motions in like circumstances.
>
> "I have formerly stated my belief that these motions of the particles neither arose from currents in the fluid containing them, nor depended on that intestine motion which may be supposed to accompany its evaporation."[1]

Brown noted that others before him had made similar observations in special cases. For example, a Dr. James Drummond had observed this fishy, erratic motion in fish eyes:

> "In 1814 Dr. James Drummond, of Belfast, published in the 7th Volume of the Transactions of the Royal Society of Edinburgh, a valuable Paper, entitled ‘On certain Appearances observed in the Dissection of the Eyes of Fishes.’
>
> "In this Essay, which I regret I was entirely unacquainted with when I printed the account of my Observations, the author gives an account of the very remarkable motions of the spicula which form the silvery part of the choroid coat of the eyes of fishes."

Today, we know that this motion, called ***Brownian motion*** in honor of Robert Brown, was due to random fluctuations in the number of water molecules bombarding the pollen grains from different directions.

Experiments showed that particles moved further in a given time interval if you raised the temperature, or reduced the size of a particle, or reduced the "viscosity" [2] of the fluid. In 1905, in a celebrated treatise entitled *The Theory of the Brownian Movement* [3], Albert Einstein developed a mathematical description which explained Brownian motion in terms of particle size, fluid viscosity, and temperature. Later, in 1923, Norbert Wiener gave a mathematically rigorous description of what is now referred to as a "stochastic process." Since that time, Brownian motion has been called a *Wiener process*, as well as a "diffusion process", a "random walk", and so on.

But Einstein wasn’t the first to give a mathematical description of Brownian motion. That honor belonged to a French graduate student who loved to gamble. His name was Louis Bachelier. Like many people, he sought to combine duty with pleasure, and in 1900 in Paris presented his doctoral thesis, entitled *Théorie de la spéculation*.

What interested Bachelier were not pollen grains and fish eyes. Instead, he wanted to know why the *prices* of stocks and bonds jiggled around on the Paris bourse. He was particularly intrigued by bonds known as *rentes sur l'état*— perpetual bonds issued by the French government. What were the laws of this jiggle? Bachelier wondered. He thought the answer lay in the prices being bombarded by small bits of news. ("The British are coming, hammer the prices down!")

The Square Root of Time

Among other things, Bachelier observed that the ***probability intervals*** into which prices fall seemed to increased or decreased with the square-root of time ($T^{0.5}$). This was a key insight.

By "probability interval" we mean a given probability for a range of prices. For example, prices might fall within a certain price range with 65 percent probability over a time period of one year. But over two years, the same price range that will occur with 65 percent probability will be larger than for one year. How much larger? Bachelier said the change in the price range was proportional to the square root of time.

Let P be the current price. After a time T, the prices will (with a given probability) fall in the range

$(P - \mathbf{a}\, T^{0.5}, P + \mathbf{a}\, T^{0.5})$, for some constant $\mathbf{a}$.

For example, if T represents one year (T=1), then the last equation simplifies to

$(P - \mathbf{a}, P + \mathbf{a})$, for some constant $\mathbf{a}$.

The price variation over two years (T=2) would be

$\mathbf{a}\, T^{0.5} = \mathbf{a}(2)^{0.5} = 1.4142\, \mathbf{a}$

or 1.4142 times the variation over one year. By contrast, the variation over a half-year (T=0.5) would be

$\mathbf{a}\, T^{0.5} = \mathbf{a}(0.5)^{0.5} = .7071\, \mathbf{a}$

or about 71 percent of the variation over a full year. That is, after 0.5 years, the price (with a given probability) would be in the range

(P –.7071**a** , P + .7071**a**).

Here the constant **a** has to be determined, but one supposes it will be different for different types of prices: **a** may be bigger for silver prices than for gold prices, for example. It may be bigger for a share of Yahoo stock than for a share of IBM.

The range of prices for a given probability, then, depends on the constant a, and on the square root of time ($T^{0.5}$). This was Bachelier's insight.

Normal Versus Lognormal

Now, to be sure, Bachelier made a financial mistake. Remember (from Part 1 of this series) that in finance we always take logarithms of prices. This is for many reasons. Most changes in most economic variables are *proportional* to their current level. For example, it is plausible to think that the variation in gold prices is proportional to the level of gold prices: $800 dollar gold varies in greater increments than does gold at $260.

The change in price, ΔP, as a proportion of the current price P, can be written as:

$\Delta P/P$.

But this is approximately the same as the change in the log of the price:

$\Delta P/P \approx \Delta (\log P)$.

What this means is that Bachelier should have written his equation:

($\log P - \mathbf{a}\, T^{0.5}$, $\log P + \mathbf{a}\, T^{0.5}$), for some constant **a**.

However, keep in mind that Bachelier was making innovations in both finance and in the mathematical theory of Brownian motion, so he had a hard enough time getting across the basic idea, without worrying about fleshing out all the correct details for a non-existent reading audience. And, to be sure, almost no one read Bachelier's PhD thesis, except the celebrated mathematician Henri Poincaré, one of his instructors.

The range of prices for a given probability, then, depends on the constant **a**, and on the square root of time ($T^{0.5}$), ***as well as the current price level P***.

To see why this is true, note that the probability range for the *log* of the price

(log P $-\mathbf{a}\ T^{0.5}$, log P + $\mathbf{a}\ T^{0.5}$)

translates into a probability range for the *price itself* as

(P exp(- $\mathbf{a}\ T^{0.5}$), P exp($\mathbf{a}\ T^{0.5}$)) .

(Here "exp" means exponential, remember? For example, exp(-.7) = $e^{-.7}$ = $2.718281^{-.7}$ = .4966.)

Rather than adding a plus or minus something to the current price P, we multiply something by the current price P. So the answer depends on the level of P. For a half-year (T=0.5), instead of

(P –.7071**a** , P + .7071**a**)

we get

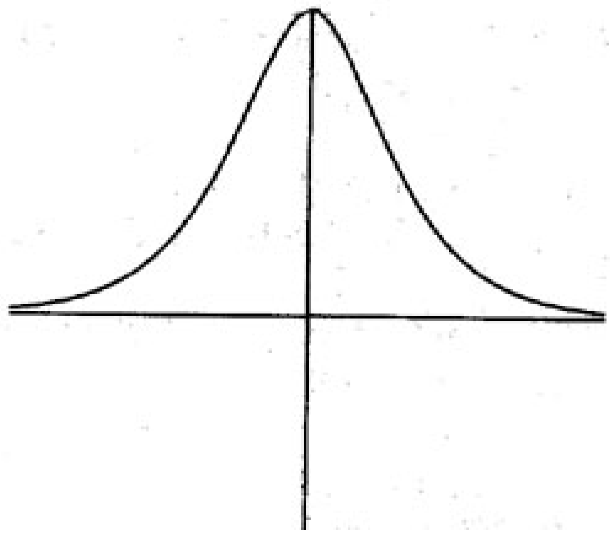
(P exp(- .7071 **a**), P exp(.7071 **a**)) .

The first interval has a constant width of 1.4142 **a**, no matter what the level of P (because P + .7071 **a** - (P -.7071 **a**) = 1.4142 **a**). But the width of the second interval varies as P varies. If we double the price P, the width of the interval doubles also.

Bachelier allowed the price range to depend on the constant **a** and on the square root of time ($T^{0.5}$), but omitted the requirement that the range should also depend on the current price level P.

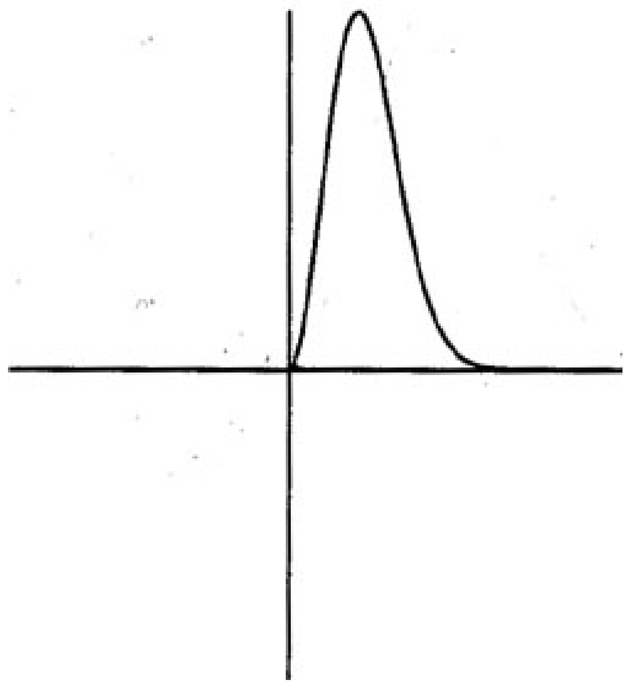
The difference in the two approaches is that if price increments (ΔP) are independent, and have a finite variance, then the price P has a ***normal*** (Gaussian distribution). But if increments in the log of the price (Δ log P) are independent, and have a finite variance, then the price P has a ***lognormal*** distribution.

Here is a picture of a normal or Gaussian distribution:



The left-hand tail never becomes zero. No matter where we center the distribution (place the mean), there is always positive probability of negative numbers.

Here is a picture of a lognormal distribution:



The left-hand tail of a lognormal distribution becomes zero at zero. No matter where we center the distribution (place the mean), there is zero probability of negative numbers.

A lognormal distribution assigns zero probability to negative prices. This makes us happy because most businesses don't charge negative prices. (However, US Treasury bills paid negative interest rates on certain occasions in the 1930s.) But a normal distribution assigns positive probability to negative prices. We don't want that.

So, at this point, we have seen Bachelier's key insight that probability intervals for prices change proportional to the square root of time (that is, the probability interval around the current price P changes by **a** $T^{0.5}$), and have modified it slightly to say that probability intervals for the *log of prices* change proportional to the square root of time (that is, the probability interval around log P changes by **a** $T^{0.5}$).

How Big Is It?

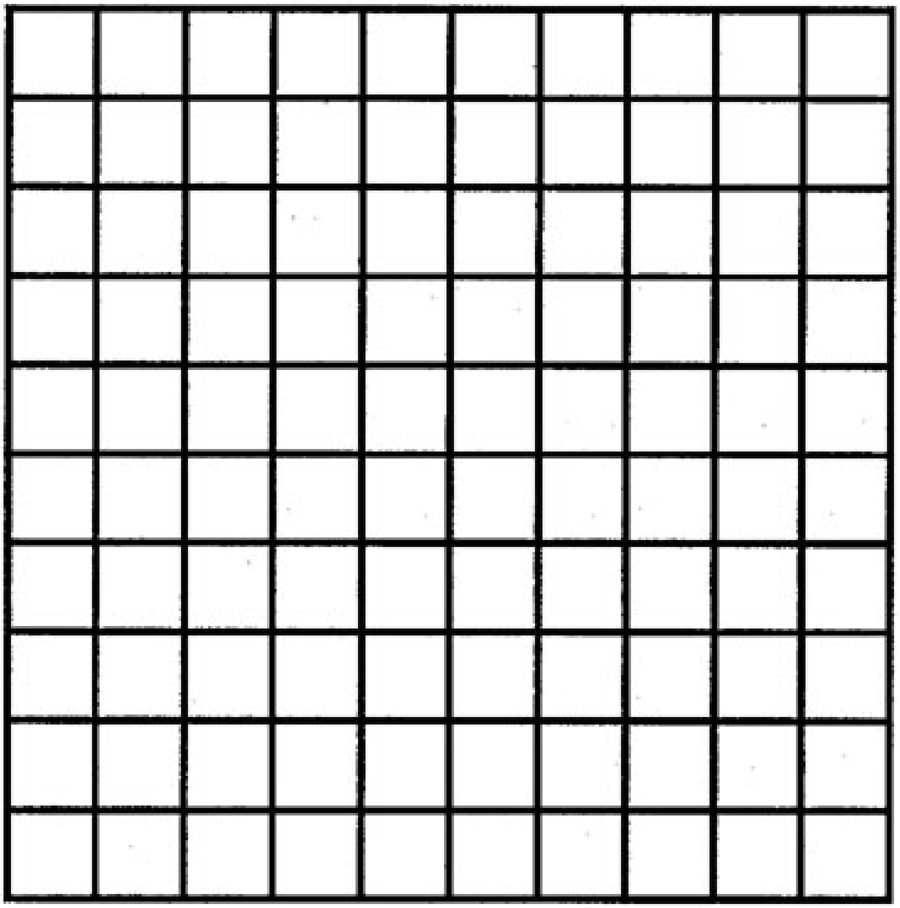
Now we are going to take a break from price distributions, and pursue the question of how we measure things. How we measure length, area, volume, or time. (This will lead us from Bachelier to Mandelbrot.)

Usually, when we measure things, we use everyday dimensions (or at least the ones we are familiar with from elementary plain geometry). A point has zero dimension. A line has one dimension. A plane or a square has two dimensions. A cube has three dimensions. These basic, common-sense type dimensions are sometimes referred to as ***topological dimensions***.

We say a room is so-many "square feet" in size. In this case, we are using the two-dimensional concept of area. We say land is so-many "acres" in size. Here, again, we are using a two-dimensional concept of area, but with different units (an "acre" being 43,560 "square feet"). We say a tank holds so-many "gallons". Here we are using a measure of volume (a "gallon" being 231 "cubic inches" in the U.S., or .1337 "cubic feet").

Suppose you have a room that is 10 feet by 10 feet, or 100 square feet. How much carpet does it take to cover the room? Well, you say, a 100 square feet of carpet, of course. And that is true, for ordinary carpet.

Let's take a square and divide it into smaller pieces.
Let's divide each side by 10:



We get 100 pieces. That is, if we divide by a scale factor of 10, we get 100 smaller squares, all of which look like the big square. If we multiply any one of the smaller squares by 10, we get the original big square.

Let's calculate a dimension for this square. We use the same formula as we used for the Sierpinski carpet:

$N = r^D$.

Taking logs, we have log N = D log r, or D = log N/ log r.

We have N = 100 pieces, and r = 10, so we get the dimension D as

D = log(100)/log(10) = 2.

(We are using "log" to mean the natural log, but notice for this calculation, which involves the ratio of two logs, that it doesn't matter what base we use. You can use logs to the base 10, if you wish, and do the calculation in your head.)

We called the dimension D calculated in this way (namely**, by comparing the number of similar objects N we got at different scales to the scale factor r**) a *Hausdorff dimension*. In this case, the Hausdorff dimension 2 is the same as the ordinary or topological dimension 2.

So, in any case, the dimension is 2, just as you suspected all along. ***But suppose you covered the floor with Sierpinski carpet. How much carpet do you need then?***

We saw (in Part 1) that the Sierpinski carpet had a Hausdorff dimension D = 1.8927… A Sierpinski carpet which is 10 feet on each side would only have $N = 10^{1.8927} =$ 78.12 square feet of material in it.

Why doesn't a Sierpinski carpet with 10 feet on each side take 100 square feet of material? Because the Sierpinski carpet *has holes in it*, of course.

Remember than when we divided the side of a Sierpinski carpet by 3, we got only 8 copies of the original because we threw out the center square. So it had a Hausdorff dimension of D = log 8/ log 3 = 1.8927. Then we divided each of the 8 copies by 3 again , threw out the center squares once more, leaving 64 copies of the original. Dividing by 3 twice is the same as dividing by 9, so, recalculating our dimension, we get D = log 64/ log 9 = 1.8927.

An ordinary carpet has a Hausdorff dimension of 2 and a topological (ordinary) dimension of 2. A Sierpinski carpet has a Hausdorff dimension of 1.8927 and a topological dimension of 2. [4]

Benoit Mandelbrot defined a ***fractal*** as **an object whose Hausdorff dimension is different from its topological dimension**. So a Sierpinski carpet is a fractal. An ordinary carpet isn't.

Fractals are cheap and sexy. A Sierpinski carpet needs only 78.12 square feet of material to cover 100 square feet of floor space. Needing less material, a Sierpinski carpet costs less. Sure it has holes in it. But the holes form a really neat pattern. So a Sierpinski carpet is sexy. Cheap and sexy. You can't beat that.

History's First Fractal

Let's see if we have this fractal stuff straight. Let's look at the first known fractal, created in 1870 by the mathematical troublemaker George Cantor.

Remember that we create a fractal by forming similar patterns at different scales, as we did with the Sierpinski carpet. It's a holey endeavor. In order to get a carpet whose Hausdorff dimension was less than 2, we created a pattern of holes in the carpet. So we ended up with an object whose Hausdorff dimension D (which compares the number N of different, but similar, objects at different scales r, $N = r^D$) was more than 1 but less than 2. That made the Sierpinski carpet a fractal, because its Hausdorff dimension was different from its topological dimension.

What George Cantor created was an object whose dimension was more than 0 but less than 1. That is, a holey object that was more than a point (with 0 dimensions) but less than a line (with 1 dimension). It's called Cantor dust. When the Cantor wind blows, the dust gets in your lungs and you can't breathe.

To create Cantor dust, draw a line and cut out the middle third:

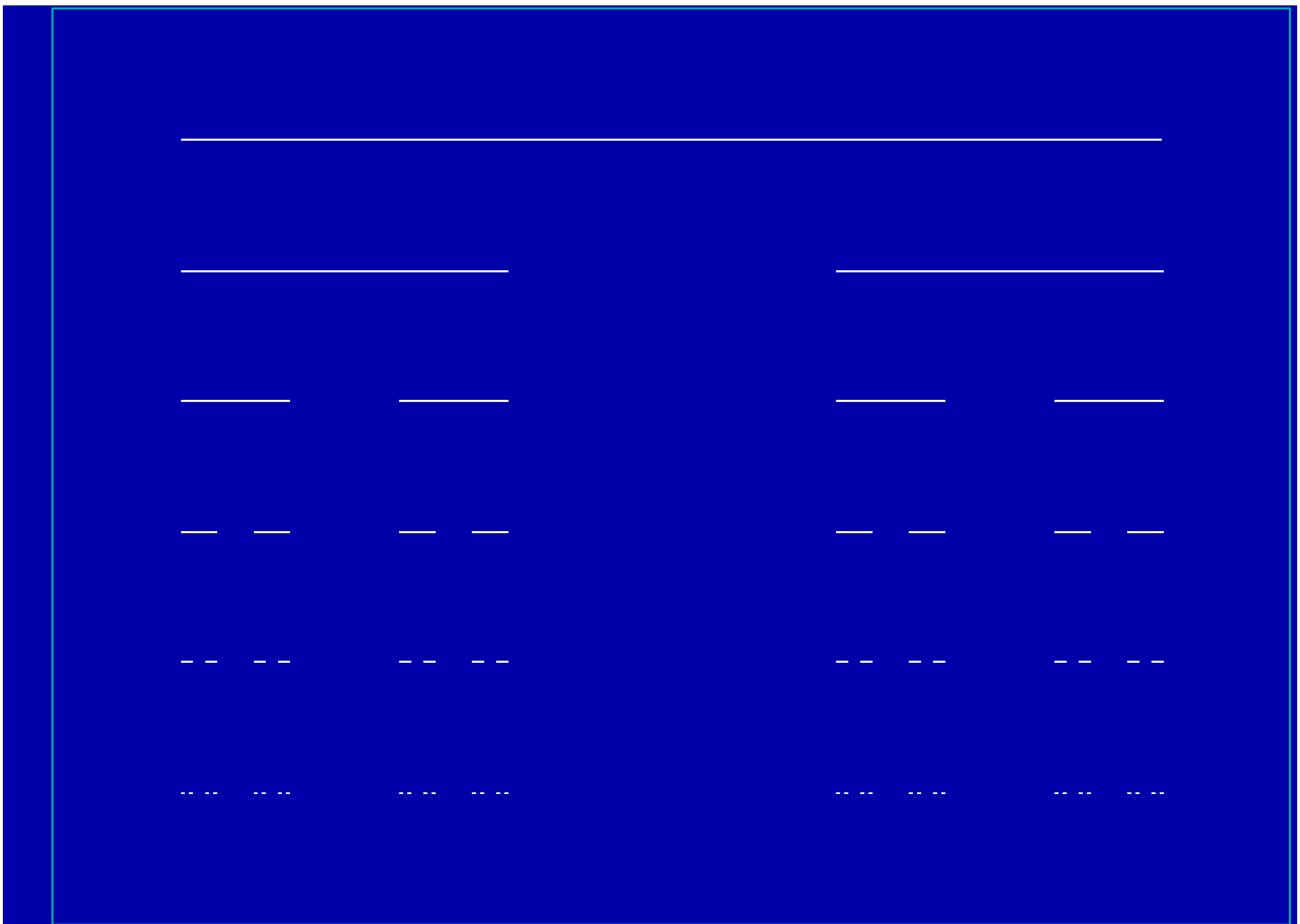
```
0____________________________________________________1
```

```
0__________________1/3                   2/3_________________1
```

Now cut out the middle thirds of each of the two remaining pieces:

```
0____1/9   2/9____ 1/3                   2/3____7/9  8/9 ____1
```

Now cut out the middle thirds of each of the remaining four pieces, and proceed in this manner for an infinite number of steps, as indicated in the following graphic.



Cantor Set (Dimension = 0.631)

What's left over after all the cutting is *Cantor dust*.

At each step we changed *scale* by r = 3, because we divided each remaining part into 3 pieces. (Each of these pieces had 1/3 the length of the original part.) Then we threw away the middle piece. (That’s how we created the holes.) That left 2 pieces. At the next step there were 4 pieces, then 8, and so on. At each step *the number of pieces* increased by a factor of N = 2. Thus the Hausdorff dimension for Cantor dust is:

$D = \log 2 / \log 3 = .6309$.

Is Cantor dust a fractal? Yes, as long as the topological dimension is different from .6309, which it surely is.

But—what is the topological dimension of Cantor dust? We can answer this by seeing how much of the original line (with length 1) we cut out in the process of making holes.

At the first step we cut out the middle third, or a length of 1/3. The next step we cut out the middle thirds of the two remaining pieces, or a length of 2(1/3)(1/3). And so on. The total length cut out is then:

$1/3 + 2(1/3^2) + 4(1/3^3) + 8(1/3^4) + \ldots = 1$.

We cut out all of the length of the line (even though we left an infinite number of points), so the Cantor dust that's left over has length zero. Its topological dimension is zero. Cantor dust is a fractal with a Hausdorff dimension of .6309 and a topological dimension of 0.

Now, the subhead refers to Cantor dust as "history’s first fractal". That a little anthropocentric. Because nature has been creating fractals for millions of years. In fact, most things in nature are not circles, squares, and lines. Instead they are fractals, and the creation of these fractals are usually determined by chaos equations. Chaos and fractal beauty are built into the nature of reality. Get used to it.

> Today, there are roughly of order 10^3 recognized fractal systems in nature, though a decade ago when Mandelbrot's classic *Fractal Geometry of Nature* was written, many of these systems were not known to be fractal. [5]

Fractal Time

So far we’ve seen that measuring things is a complicated business. Not every length can be measured with a tape measure, nor the square footage of material in every carpet measured by squaring the side of the carpet.

Many things in life are fractal, and follow power laws just like the D of the Hausdorff dimension. For example, the "loudness" L of noise as heard by most humans is proportional to the sound intensity I raised to the fractional power 0.3:

$L = a\ I^{0.3}$.

Doubling the loudness at a rock concert requires increasing the power output by a factor of ten, because

$a\ (10\ I)^{0.3} = 2\ a\ I^{0.3} = 2\ L$.

In financial markets, another subjective domain, "time" is fractal. Time does not always move with the rhythms of a pendulum. Sometimes time is less than that. In fact, we've already encounted fractal time with the Bachelier process, where the log of probability moved according to

$\mathbf{a}\ T^{0.5}$.

Bachelier observed that if the time interval was multiplied by 4, the probability interval only increased by 2. In other words, at a scale of **r** = 4, the number N of similar probability units was N = 2. So the Hausdorff dimension for time was:

$D = \log N / \log r = \log 2 / \log 4 = 0.5$.

In going from Bachelier to Mandelbrot, then, the innovation is not in the observation that time is fractal: that was Bachelier's contribution. Instead the question is: What is the correct fractal dimension for time in speculative markets? Is the Hausdorff dimension really D = 0.5, or does it take other values? And if the Hausdorff dimension of time takes other values, what's the big deal, anyway?

The way in which Mandlebrot formulated the problem provides a starting point:

> Despite the fundamental importance of Bachelier's process, which has come to be called "Brownian motion," it is now obvious that it does not account for the abundant data accumulated since 1900 by empirical economists, simply because *the empirical distributions of price changes are usually too "peaked" to be relative to*

samples from Gaussian populations. [6]

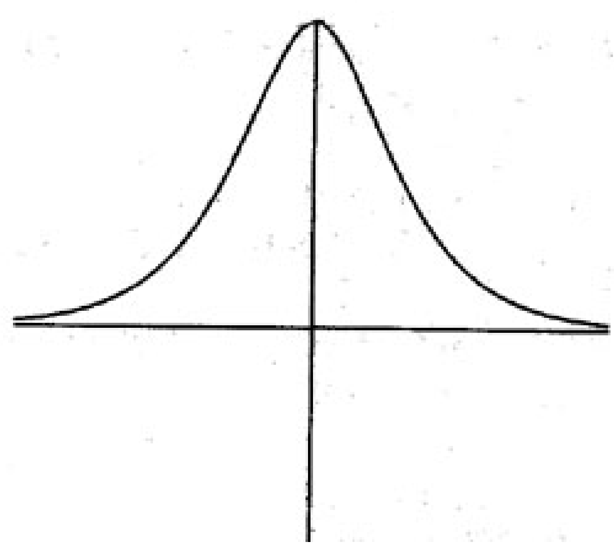
What does Mandelbrot mean by "peaked"? It's now time for a discussion of probability.

Probability is a One-Pound Jar of Jelly

Probability is a one-pound jar of jelly. You take the jelly and smear it all over the real line. The places where you smear more jelly have more probability, while the places where you smear less jelly have less probability. Some spots may get no jelly. They have no probability at all—their probability is zero.

The key is that you *only have one pound* of jelly. So if you smear more jelly (probability) at one location, you have to smear less jelly at another location.

Here is a picture of jelly smeared in the form of a bell-shaped curve:



The jelly is smeared between the horizontal (real) line all the way up to the curve, with a uniform thickness. The result is called a "standard normal distribution". ("Standard" because its mean is 0, and the standard deviation is 1.) In this picture, the point where the vertical line is and surrounding points have the jelly piled high—hence they are more probable.

As we observed previously, for the normal distribution jelly gets smeared on the real (horizontal) line all the way to plus or minus infinity. There may not be much jelly on the distant tails, but there is always some.

Now, let's think about this bell-shaped picture. What does Mandelbrot mean by the distribution of price changes being "too peaked" to come from a normal distribution?

Does Mandelbrot's statement make any sense? If we smear more jelly at the center of the bell curve, to make it taller, we can only do so by taking jelly from some other place. Suppose we take jelly out of the tails and intermediate parts of the distribution and pile it on the center. The distribution is now "more peaked". It is more centered in one place. It has a smaller standard deviation—or smaller dispersion around the mean. But—it could well be still normal.

So what's with Mandelbrot, anyway? What does he mean? We'll discover this in Part 3 of this series.

Click here to see the Answer to Problem 1 from Part 1. The material therein should be helpful in solving **Problem 2**.

Meanwhile, here are two new problems for eager students:

Problem 3: Suppose you create a Cantor dust using a different procedure. Draw a line. Then divide the line into 5 pieces, and throw out the second and fourth pieces. Repeat this procedure for each of the remaining pieces, and so on, for an infinite number of times. What is the fractal dimension of the Cantor dust created this way? What is its topological dimension? Did you create a new fractal?

Problem 4: Suppose we write all the numbers between 0 and 1 in ternary. (Ternary uses powers of 3, and the numbers 0, 1, 2. The ternary number .1202, for example, stands for 1 x 1/3 + 2 x 1/9 + 0 x 1/27 + 2 x 1/81.) Show the Cantor dust we created here in Part 2 (with a Hausdorff dimension of .6309) can be created by taking all numbers between 0 and 1, and eliminating those numbers whose ternary expansion contains a 1. (In other words, what is left over are all those numbers whose ternary expansions only have 0s and 2s.)

And enjoy the fractal:

Notes

[1] Robert Brown, "Additional Remarks on Active Molecules," 1829.

[2] Viscosity is a fluid’s stickiness: honey is more viscous than water, for example. "Honey don’t jiggle so much."

[3] I am using the English title of the well-known Dover reprint: *Investigations on the Theory of the Brownian Movement*, Edited by R. Furth, translated by A.D. Cowpter, London, 1926. The original article was in German and titled somewhat differently.

[4] I am admittingly laying a subtle trap here, because of the undefined nature of "topological dimension". This is partially clarified in the discussion of Cantor dust, and further discussed in Part 3.

[5] H. Eugene Stanley, *Fractals and Multifractals*, 1991

[6] Benoit Mandelbrot, "The Variation of Certain Speculative Prices," *Journal of Business*, 36(4), 394-419, 1963.

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from *The Laissez Faire City Times*, Vol 3, No 24, June 14, 1999

Chaos and Fractals in Financial Markets

Part 6

by J. Orlin Grabbe

Prechter's Drum Roll

Robert Prechter is a drummer. He faces the following problem. He wants to strike his drum three times, creating two time intervals which have a special ratio:

1<------------***g***-------------->2<--------------------***h***-------------------->3

Here is the time ratio he is looking for: he wants the ratio of the first time interval to the second time interval to be the same as the ratio of the second time interval to the entire time required for the three strikes.

Let the first time internal (between strikes 1 and 2) be labeled g, while the second time interval (between strikes 2 and 3) be labeled h. So what Prechter is looking for is the ratio of g to h to be the same as h to the whole. However, the whole is simply $g + h$, so Prechter seeks g and h such that:

$g / h = h / (g+h)$.

Now. Prechter is only looking for a particular ratio. He doesn't care whether he plays his drum slow or fast. So h can be anything: 1 nano-second, 1 second, 1 minute, or whatever. So let's set $h = 1$. (Note that by setting $h = 1$, we are choosing our **unit of measurement**.) We then have

$g / 1 = 1 / (1+g)$.

Multiplying the equation out we get

$g^2 + g - 1 = 0$.

This gives two solutions:

$g = [- 1 + 5^{0.5}] / 2 = 0.618033\ldots$, and

$g = [- 1 - 5^{0.5}] / 2 = -1.618033\ldots$

The first, positive solution **($g = 0.618033\ldots$)** is called the **golden mean**. Using $h = 1$ as our scale of measurement, then g, the golden mean, is the solution to the ratio

g / h = ***h / (g+h)***.

By contrast, if we use ***g*** = 1 as our scale of measurement, and solve for ***h***, we have

1 / *h* = ***h / (*1+*h)***, which gives the equation

$$h^2 - h - 1 = 0.$$

Which gives the two solutions:

$h = [\,1 + 5^{0.5}] / 2 = 1.618033\ldots$, and

$h = [\,1 - 5^{0.5}] / 2 = -0.618033\ldots$

Note that since the units of measurement are somewhat aribitrary, ***h*** has as much claim as ***g*** to being the solution to Prechter's drum roll. Naturally, ***g*** and ***h*** are closely related:

h (using ***g*** as the unit scale) = 1/ ***g*** **(**using ***h*** as the unit scale).

for either the positive or negative solutions:

1.618033… = 1/ 0.618033…

-0.618033… = 1/ -1.618033.

What is the meaning of the negative solutions? These also have a physical meaning, depending on where we place our ***time origin***. For example, let's let the second strike of the drum be time **t**=0:

<------------***g***-------------->0<-------------------***h***-------------------->

Then we find that for ***g*** = -1.618033, ***h*** = 1, we have

-1.618033 /1 = 1/ [1 - 1.618033].

So the negative solutions tell us the same thing as the positive solutions; but they correspond to a time origin of **t** = 0 for the second strike of the drum.

The same applies for ***g*** = 1, ***h*** = -0.618033, since

1 / -0.618033 = -0.618033/(1 – 0.618033),

but in this case **time is running backwards**, not forwards.

The golden mean ***g***, or its reciprocal equivalent ***h***, are found throughout the natural world. Numerous books have been devoted to the subject. These same ratios are found in financial markets.

Symmetric Stable Distributions and the Golden Mean Law

In Part 5, we saw that symmetric stable distributions are a type of probability distribution that are fractal in nature: a sum of **n** independent copies of a symmetric stable distribution is related to each copy by a scale factor $\mathbf{n^{1/\alpha}}$**,** where α is the Hausdorff dimension of the given symmetric stable distribution.

In the case of the normal or Gaussian distribution, the Hausdorff dimension α **= 2**, which is equivalent to the dimension of a plane. A Bachelier process, or Brownian motion (as first covered in Part 2), is governed by a $T^{1/\alpha} = T^{1/2}$ law.

In the case of the Cauchy distribution (Part 4), the Hausdorff dimension α **= 1**, which is equivalent to the dimension of a line. A Cauchy process would be governed by a $T^{1/\alpha} = T^{1/1} = T$ law.

In general, $0 < \alpha <=2$. This means that *between* the Cauchy and the Normal are all sorts of interesting distributions, including ones having the same Hausdorf dimension as a Sierpinski carpet (α = log 8/ log 3 = 1.8927….) or Koch curve (α = log 4/ log 3 = 1.2618….).

Interestingly, however, many financial variables are symmetric stable distributions with an α parameter that hovers around the value of ***h*** = 1.618033, where ***h*** is the reciprocal of the golden mean ***g*** derived and discussed in the previous section. This implies that these market variables follow a time scale law of $T^{1/\alpha} = T^{1/h} = T^{\mathbf{g}} = T^{\mathbf{0.618033..}}$. **That is, these variables following a T-to-the-golden-mean power law, by contrast to Brownian motion, which follows a T-to-the-one-half power law.**

For example, I estimated α for daily changes in the dollar/deutschemark exchange rate for the first six years following the breakdown of the Bretton Woods Agreement of fixed exchange rates in 1973. [1] (The time period was July 1973 to June 1979.) The value of α was calculated using maximum likelihood techniques [2]. The value I found was

$\alpha = 1.62$

with a margin of error of plus or minus .04. You can’t get much closer than that to α **=** ***h*** **=** 1.618033…

In this and other financial asset markets, it would seem that time scales not according to the commonly assumed square-root-of-T law, but rather to a $T^{\mathbf{g}}$ law.

The Fibonacci Dynamical System

The value of ***h* =** 1.618033… is closely related to the ***Fibonacci*** sequence of numbers. The Fibonacci sequence of numbers is a sequence in which each number is the sum of the previous two:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

Notice the third number in the sequence, 2=1+1. The next number 3=2+1. The next number 5=3+2. And so on, each number being the sum of the two previous numbers.

This mathematical sequence appeared 1202 A.D. in the book *Liber Abaci*, written by the Italian mathematician Leonardo da Pisa, who was popularly known as Fibonacci (son of Bonacci). Fibonacci told a story about rabbits. These were *mathematical rabbits* that live forever, take one generation to mature, and always thereafter have one off-spring per generation. So if we start with 1 rabbit (the first 1 in the Fibonaaci sequence), the rabbit takes one generation to mature (so there is still 1 rabbit the next generation—the second 1 in the sequence), then it has a baby rabbit in the following generation (giving 2 rabbits—the 2 in the sequence), has another offspring the next generation (giving 3 rabbits); then, in the next generation, the first baby rabbit has matured and also has a baby rabbit, so there are two offspring (giving 5 rabbits in the sequence), and so on.

Now, the Fibonacci sequence represents the path of a dynamical system. We introduced dynamical systems in Part 1 of this series. (In Part 5, we discussed the concept of Julia Sets, and used a particular dynamical system—the complex logistic equation—to create computer art in real time using Java applets. The Java source code was also included.)

The Fibonacci dynamical system look like this:

$F(n+2) = F(n+1) + F(n)$.

The number of rabbits in each generation ($F(n+2)$) is equal to the sum of the rabbits in the previous two generations (represented by $F(n+1)$ and $F(n)$). This is an example of a more general dynamical system that may be written as:

$F(n+2) = p\, F(n+1) + q\, F(n)$,

where p and q are some numbers (parameters). The solution to the system depends on the values of p and q, as well as the starting values $F(0)$ and $F(1)$. For the Fibonacci system, we have the simplification $p = q = F(0) = F(1) = 1$.

I will not go through the details here, but the Fibonacci system can be solved to yield the solution:

$F(n) = [1/5^{0.5}] \{ [(1+5^{0.5})/2]^n – [(1-5^{0.5})/2]^n \}, n = 1, 2, . . .$

The following table gives the value of F(n) for the first few values of n:

n	1	2	3	4	5
F(n)	1	1	2	3	5

And so on for the rest of the numbers in the Fibonacci sequence. Notice that the general solution involves the two solution values we previously calculated for ***h*.** To simplify, however, we will now write everything in terms of the first of these values (namely, ***h*** = 1.618033 …). Thus we have

$\boldsymbol{h} = [1 + 5^{0.5}] / 2 = 1.618033…$, and

$\mathbf{- 1/} \boldsymbol{h} = [1 - 5^{0.5}] / 2 = -0.618033…$

Inserting these into the solution for the Fibonacci system F(n), we get

$F(n) = [1/5^{0.5}] \{ [\boldsymbol{h}]^n – [\mathbf{-1/}\boldsymbol{h}]^n \}, n = 1, 2, . . .$

Alternatively, writing the solution using the golden mean ***g***, we have

$F(n) = [1/5^{0.5}] \{ [\boldsymbol{g}]^{-n} – [-\boldsymbol{g}]^n \}, n = 1, 2, . . .$

The use of Fibonacci relationships in financial markets has been popularized by Robert Prechter [3] and his colleagues, following the work of R. N. Elliott [4]. The empirical evidence that the Hausdorff dimension of some symmetric stable distributions encountered in financial markets is approximately $\alpha = \boldsymbol{h} =$ 1.618033… indicates that this approach is based on a solid empirical foundation.

Notes

[1] See "Research Strategy in Empirical Work with Exchange Rate Distributions," in J. Orlin Grabbe, *Three Essays in International Finance*, Ph.D. Thesis, Department of Economics, Harvard University, 1981.

[2] There are two key papers by DuMouchel which yield the

background needed for doing maximum likelihood estimates of α, where $\alpha < 2$:

DuMouchel, William H. (1973), "On the Asymptotic Normality of the Maximum Likelihood Estimate when Sampling from a Stable Distribution," *Annals of Statistics*, 1, 948-57.

DuMouchel, William H. (1975), "Stable Distributions in Statistical Inference: 2. Information from Stably Distributed Samples," *Journal of the American Statistical Association*, 70, 386-393.

[3] See, for example:

Robert R. Prechter, Jr., *At the Crest of the Tidal Wave*, John Wiley & Sons, New York, 1995

Robert R. Prechter, Jr., *The Wave Principle of Human Social Behavior and the New Science of Socionomics*, New Classics Library, Gainesville, Georgia, 1999.

[4] See *R.N. Elliott's Masterworks—The Definitive Collection*, edited by Robert R. Prechter, Jr., Gainesville, Georgia, 1994.

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from *The Laissez Faire City Times*, Vol 3, No 35, September 6, 1999

Chaos and Fractals in Financial Markets

Part 5

by J. Orlin Grabbe

Louis Bachelier Visits the New York Stock Exchange

Louis Bachelier, resurrected for the moment, recently visited the New York Stock Exchange at the end of May 1999. He was somewhat puzzled by all the hideous concrete barriers around the building at the corner of Broad and Wall Streets. For a moment he thought he was in Washington, D.C., on Pennsylvania Avenue.

Bachelier was accompanied by an angelic guide named Pete. "The concrete blocks are there because of Osama bin Ladin," Pete explained. "He's a terrorist." Pete didn't bother to mention the blocks had been there for years. He knew Bachelier wouldn't know the difference.

"Terrorist?"

"You know, a ruffian, a scoundrel."

"Oh," Bachelier mused. "Bin Ladin. The son of Ladin."

"Yes, and before that, there was Abu Nidal."

"Abu Nidal. The father of Nidal. Hey! Ladin is just Nidal spelled backwards. So we've gone from the father of Nidal to the son of backwards-Nidal?"

"Yes," Pete said cryptically. "The spooks are never too creative when they are manufacturing the boogeyman of the moment. If you want to understand all this, read about 'Goldstein' and the daily scheduled 'Two Minutes Hate' in George Orwell's book *1984*."

"1984? Let's see, that was fifteen years ago," Bachelier said. "A historical work?"

"Actually, it's futuristic. But he who controls the present controls the past, and he who controls the past controls the future."

Bachelier was mystified by the entire conversation, but once they got inside and he saw the trading floor, he felt right at home. Buying, selling, changing prices. The chalk boards were now electric, he saw, and that made the air much fresher.

"Look," Bachelier said, "the Dow Jones average is still around!"

"Yes," nodded Pete, "but there are a lot of others also. Like the S&P500 and the New York Stock Exchange Composite Index."

"I want some numbers!" Bachelier exclaimed enthusiastically. Before they left, they managed to con someone into giving them the closing prices for the NYSE index for the past 11 days.

"You can write a book," Pete said. "Call it *Eleven Days in May*. Apocalyptic titles are all the rage these days—except in the stock market."

Bachelier didn't pay him any mind. He had taken out a pencil and paper and was attempting to calculate logarithms through a series expansion. Pete watched in silence for a while, before he took pity and pulled out a pocket calculator.

"Let me show you a really neat invention," the angel said.

Bachelier's Scale for Stock Prices

Here is Bachelier's data for eleven days in May. We have the calendar date in the first column of the table; the NYSE Composite Average, S(t), in the second column; the log of S(t) in the third column; the change in log prices, x(t) = log S(t) – log S(t-1) in the fourth column; and $x(t)^2$ in the last column. The sum of the variables in the last column is given at the bottom of the table.

Date	S(t)	log S(t)	x(t)	$x(t)^2$
May 14	638.45	6.459043		
May 17	636.92	6.456644	-.002399	.000005755
May 18	634.19	6.452348	-.004296	.000018456

May 19	639.54	6.460749	.008401	.000070577
May 20	639.42	6.460561	-.000188	.000000035
May 21	636.87	6.456565	-.003996	.000015968
May 24	626.05	6.439430	-.017135	.000293608
May 25	617.34	6.425420	-.014010	.000196280
May 26	624.84	6.437495	.012075	.000145806
May 27	614.02	6.420027	-.017468	.000305131
May 28	622.26	6.433358	.013331	.000177716
sum of all $x(t)^2$ = .001229332				

What is the meaning of all this?

The variables x(t), which are the one-trading-day changes in log prices, are the variables in which Bachelier is interested for his theory of Brownian motion as applied to the stock market:

$x(t) = \log S(t) – \log S(t-1)$.

Bachelier thinks these should have a normal distribution. Recall from Part 4 that a normal distribution has a location parameter **m** and a scale parameter **c**. So what Bachelier is trying to do is to figure out what **m** and **c** are, assuming that each day's **m** and **c** are the same as any other day's.

The location parameter **m** is easy. It is zero, or pretty close to zero.

In fact, it is not quite zero. Essentially there is a drift in the movement of the stock index **S(t)**, given by the difference between the interest rate (such as the broker-dealer loan rate) and the dividend yield on stocks in the average.[1] But this is tiny over our eleven trading days (which gives us ten values for **x(t)**). So Bachelier just assumes **m** is zero.

So what Bachelier is doing with the data is trying to estimate **c**.

Recall from Part 2 that if today's price is **P**, Bachelier modeled the *probability interval* around the log of the price change by

(log P – **a** $T^{0.5}$, log P + **a** $T^{0.5}$), for some constant **a**.

But now, we are writing our stock index price as **S**, not **P**; and the constant **a** is just our scale parameter **c**. So, changing notation, Bachelier is interested in the probability interval

(log **S** – **c** $T^{0.5}$, log **S** + **c** $T^{0.5}$), for a given scale parameter **c**.

One way of estimating the scale **c (c** is also called the "standard deviation" in the context of the normal distribution) is to add up all the squared values of **x(t)**, and take the average (by dividing by the number of observations). This gives us an estimate of the variance, or $\mathbf{c}^2$. Then we simply take the square root to get the scale **c** itself. (This is called a *maximum likelihood estimator* for the standard deviation.)

Adding up the terms in the right-hand column in the table gives us a value of .001229332. And there are 10 observations. So we have

variance = $\mathbf{c}^2$ = .001229332/10 = .0001229332.

Taking the square root of this, we have

standard deviation = **c** = $(.0001229332)^{0.5}$ = .0110875.

So Bachelier's changing probability interval for log **S** becomes:

(log **S** – **.0110875** $T^{0.5}$, log **S** + **.0110875** $T^{0.5}$).

To get the probability interval for the price **S** itself, we just take exponentials (raise to the power **exp** = e = 2.718281…), and get

(**S** exp(– .0110875 $T^{0.5}$), **S** exp(.0110875 $T^{0.5}$)).

Since the current price on May 28, from the table, is 622.26, this interval becomes:

(622.26 exp(– .0110875 $T^{0.5}$), 622.26 exp(.0110875 $T^{0.5}$)).

"This expression for the probability interval tells us the probability distribution over the next T days," Bachelier explained to Pete. "Now I understand what you meant. He who controls the present controls the past, because he can obtain past data. While he who masters this past data controls the future, because he can calculate future probabilities!"

"Umm. That wasn't what I meant," the angel replied. "But never mind."

Over the next 10 trading days, we have $T^{0.5} = 10^{0.5} = 3.162277$. So substituting that into the probability interval for price, we get

(622.26 (.965545), 622.26 (1.035683)) = (600.82, 644.46).

This probability interval gives a price range for plus or minus one scale parameter (in logs) **c**. For the normal distribution, that corresponds to **68 percent probability**. With 68 percent probability, the price will lie between 600.82 and 644.46 at the end of 10 more trading days, according to this calculation.

To get a **95 percent probability** interval, we use plus or minus **2c,**

(622.26 exp(– (2) .0110875 $T^{0.5}$), 622.26 exp((2) .0110875 $T^{0.5}$)),

which gives us a price interval over 10 trading days of

(580.12, 667.46).

Volatility

In the financial markets, the scale parameter **c** is often called "volatility". Since a normal distribution is usually assumed, "volatility" refers to the standard deviation.

Here we have measured the scale **c**, or volatility, on a basis of one trading day. The value of **c** we calculated, c = .0110875, was calculated over 10 trading days, so it would be called in the markets "a 10-day historical volatility." If calculated over 30 past trading days, it would be "a 30-day historical volatility."

However, market custom would dictate two criteria by which volatility is quoted:

1. quote volatility at an *annual* (not daily) rate;
2. quote volatility in *percentage* (not decimal) terms.

To change our daily volatility **c** = .0110875 into annual terms, we note that there are about 256 trading days in the year. The square root of 256 is 16, so to change daily volatility into annual volatility, we simply multiply it by 16:

annual **c** = 16 (daily **c**) = 16 (.0110875) = .1774.

Then we convert this to percent (by multiplying by 100 and calling the result "percent"):

annual **c** = 17.74 percent.

The New York Stock Exchange Composite Index had a historical volatility of 17.74 percent over the sample period during May.

Note that an *annual* volatility of 16 percent corresponds to a *daily* volatility of 1 percent. This is a useful relationship to remember, because we can look at a price or index, mentally divide by 100, and say the price change will fall in the range of plus or minus that amount with 2/3 probability (approximately). For example, if the current gold volatility is 16 percent, and the price is $260, we can say the coming day's price change will fall in the range of plus or minus $2.60 with about 2/3 probability.

Notice that 256 trading days give us a probability interval that is only 16 times as large as the probability interval for 1 day. This translates into a Hausdorff dimension for time (in the probability calculation) as D = log(16)/log(256) = ½ or 0.5, which is just the Bachelier-Einstein square-root-of-T ($T^{0.5}$) law.

The way we calculated the scale **c** is called "historical volatility," because we used actual historical data to estimate **c**. In the options markets, there is another measure of volatility, called "**implied volatility**." Implied volatility is found by back-solving an option value (using a valuation formula) for the volatility, **c**, that gives the current option price. Hence this volatility, which pertains to the future (specifically, to the future life of the option) is *implied* by the price at which the option is trading.

Fractal Sums of Random Variables

Now for the fun part. We have been looking at random variables **x(t)** (representing changes in the log of price).

Under the assumption these random variables were normal, we estimated a scale parameter **c**, which allows us to do probability calculations.

In order to estimate **c**, we took the sums of random variables (or, in this instance, the sums of squares of x(t)).

Were our calculations proper and valid? Do they make any sense? The answer to these questions depends on the issue of *the probability distribution of a sum of random variables*. How does the distribution of the **sum** relate to the distributions of the individual random variables that are added together?

In answering this question we want to focus on ways we can come up with a location parameter **m**, and a scale parameter **c**. For the normal distribution, **m** is the *mean*, but for the Cauchy distribution the mean doesn't exist ("is infinite"). For the normal distribution, the scale parameter **c** is the standard deviation, but for the Cauchy distribution the standard deviation doesn't exist. Nevertheless, a location **m** and a scale **c** exist for the Cauchy distribution. The maximum likelihood estimator for **c** will not be the same in the case of the Cauchy distribution as it was for the normal. We can't take squares if the x(t) have a Cauchy distribution.

Suppose we have **n** random variables $\mathbf{X}_i$, *all with the same distribution*, and we calculate their sum **X**:

$\mathbf{X} = X_1 + X_2 + \ldots + X_{n-1} + X_n.$

Does the distribution of the sum **X** have a simple form? In particular, can we relate the distribution of **X** to the common distribution of the $\mathbf{X_i}$? Let's be even more specific. We have looked at the normal (Gaussian) and Cauchy distributions, both of which were parameterized with a location **m** and scale **c**. If each of the $\mathbf{X_i}$ has a location **m** and scale **c**, whether normal or Cauchy, can that information be translated into a location and a scale for the sum **X**?

The answer to all these questions is *yes*, for a class of distributions called *stable distributions*. (They are also sometimes called "Levy stable", "Pareto-Levy", or "stable Paretian" distributions.) Both the *normal* and the *Cauchy* are stable distributions. But there are many more.

We will use the notation "~" as shorthand for "has the same distribution as." For example,

$X_1 \sim X_2$

means X_1 and X_2 have the same distribution. We now use "~" in the following definition of stable distributions:

Definition: A random variable X is said to have a **stable distribution** if for any **n** >= 2 (greater than or equal to 2), there is a positive number C_n and a real number D_n such that

$X_1 + X_2 + \ldots + X_{n-1} + X_n \boldsymbol{\sim} \mathbf{C_n} X + \mathbf{D_n}$

where $X_1, X_2, \ldots, X_n$ are all independent copies of X.

Think of what this definition means. If their distribution is stable, then the sum of **n** identically distributed random variables has the same distribution as any one of them, except by multiplication by a scale factor C_n and a further adjustment by a location D_n .

Does this remind you of fractals? Fractals are geometrical objects that look the same at different scales. Here we have random variables whose probability distributions look the same at different scales (except for the add factor D_n).

Let's define two more terms.[2]

Definition: A stable random variable X is **strictly stable** if $D_n = 0$.

So strictly stable distributions are clearly* fractal *in nature, because the sum of n independent copies of the underlying distribution looks exactly the same as the

underlying distribution itself, once adjust by the scale factor C_n. One type of strictly stable distributions are *symmetric* stable distributions.

Definition: A stable random variable X is **symmetric stable** if its distribution is symmetric—that is, if **X** and **-X** have the same distribution.

The scale parameter C_n necessarily has the form [3]:

$$C_n = n^{1/\alpha}, \text{ where } 0 < \alpha <= 2.$$

So if we have **n** independent copies of a symmetric stable distribution, their sum has the same distribution with a scale that is $n^{1/\alpha}$ times as large.

For the normal or Gaussian distribution, $\alpha = 2$. So for **n** independent copies of a normal distribution, their sum has a scale that is $n^{1/\alpha} = n^{1/2}$ times as large.

For the Cauchy distribution, $\alpha = 1$. So for **n** independent copies of a Cauchy distribution, their sum has a scale that is $n^{1/\alpha} = n^{1/1} = n$ times as large.

Thus if, for example, Brownian particles had a Cauchy distribution, they would scale not according to a $T^{0.5}$ law, but rather according to a **T** law!

Notice that we can also calculate a Hausdorff dimension for symmetric stable distributions. If we divide a symmetric stable random variable **X** by a scale factor of $c = n^{1/\alpha}$, we get the probability equivalent [4] of N = **n** copies of $X/n^{1/\alpha}$. So the Hausdorff dimension is

$$D = \log N / \log c = \log n / \log(n^{1/\alpha}) = \alpha .$$

This gives us a simple interpretation of α. The parameter α is simply the Hausdorff dimension of a symmetric stable distribution. For the normal, the Hausdorff dimension is equal to 2, equivalent to that of a plane. For the Cauchy, the Hausdorff dimension is equal to 1, equivalent to that of a line. In between is a full range of values, including symmetric stable distributions with Hausdorff dimensions equivalent to the Koch Curve (log 4/log 3) and the Sierpinski Carpet (log 8/log3).

Some Fun with Logistic Art

Now that we've worked our way to the heart of the matter, let's take a break from probability theory and turn our attention again to dynamical systems. In particular, let's look at our old friend the logistic equation:

x(n+1) = **k** x(n) [1 – x(n)],

where x(n) is the input variable, x(n+1) is the output variable, and **k** is a constant.

In Part 1, we looked at a particular version of this equation where k = 4. In general, **k** takes values 0 < k <= 4.

The dynamic behavior of this equation depends on the value **k**, and also on the particular starting value or starting point, **x(0)**. Later in this series we will examine how the behavior of this equation changes as we change **k**. But not now.

Instead, we are going to look at this equation when we substitute for **x**, which is a real variable, a complex variable **z**:

z(n+1) = k z(n) [1 – z(n)].

Complex numbers z have the form

z = x + **i** y,

where **i** is the square root of minus one. Complex numbers are normally graphed in a plane, with **x** on the horizontal ("real") axis, while **y** is on the vertical ("imaginary") axis.

That means when we iterate **z**, we actually iterate *two* values: **x** in the horizontal direction, and **y** in the vertical direction. The complex logistic equation is:

x + i y = k (x + i y) [1 – (x + i y)].

(Note that I have dropped the notation x(n) and y(n) and just used **x** and **y**, to make the equations easier to read. But keep in mind that **x** and **y** on the left-hand side of the equation represent *output*, while the **x** and **y** on the right-hand side of the equation represent *input*.)

The output **x**, the *real* part of **z**, is composed of all the terms that do not multiply **i**, while the output **y**, the *imaginary* part of **z**, is made up of all the terms that multiply **i**.

To complete the transformation of the logistic equation, we let **k** be complex also, and write

k = A + B i,

giving as our final form:

x + i y = (A + B i) (x + i y) [1 – (x + i y)].

Now we multiply this all out and collect terms. The result is two equations in x and y:

$x = A (x-x^2+y^2) + B (2xy-y)$
$y = B (x-x^2+y^2) - A (2xy-y).$

As in the real version of the logistic equation, the behavior of the equation depends on the multiplier k = A + B i (that is, on A and B), as well as the initial starting value of z = x + i y (that is, on x(0) and y(0)).

Julia Sets

Depending on **k**, some beginning values z(0) = x(0) + **i** y(0) blow off to infinity after a certain number of iterations. That is, the output values of **z** keep getting larger and larger, diverging to infinity. As **z** is composed of both an **x** term and a **y** term, we use as the criterion for "getting large" the value of

$x^2 + y^2.$

The square root of this number is called the *modulus* of **z**, and represents the length of a vector from the origin (0,0) to the point **z** = (x,y). In the iterations we are about to see, the criterion to determine if the equation is diverging to infinity is

$x^2 + y^2 > 4,$

which implies the modulus of **z** is greater than 2.

When the equation is iterated, some starting values diverge to infinity and some don't. **The *Julia set* is the set of starting values for z that remain finite under iteration**. That is, the Julia set is the set of all starting values (x(0), y(0)) such that the equation output does not blow off to infinity as the equation is iterated.

Each value for **k** will produce a different Julia set (i.e., a different set of (x(0) ,y(0)) values that do not diverge under iteration).

Let's do an example. Let **k** = 1.678 + .95 **i**. That is, A = 1.678 and B = .95. We let starting values for x(0) range from –.5 to 1.5, while letting starting values for y(0) range from -.7 to +.7.

We keep **k** constant always, so we are graphing the Julia set associated with **k** = 1.678 + .95 **i**.

We iterate the equation 256 times. If, at the end of 256 iterations, the modulus of **z** is not greater than 2, we paint the starting point (x(0), y(0)) black. So the entire **Julia set in this example is colored black**. If the modulus of **z** exceeds 2 during the iterations, the starting point (x(0), y(0)) is ***assigned a color depending on the rate the equation is blowing off to infinity***.

To see the demonstration, be sure Java is enabled on your web browser and click here.

We can create a plot that looks entirely different by making a different color assignment. For the next demonstration, we again iterate the dynamical system 256 times for different starting values of **z(n)**. If, during the iterations, the modulus of **z** exceeds 2, then we know the iterations are diverging, so we plot the starting value **z(0) = (x(0), y(0))** black. Hence the **black region of the plot is made up of all the points *not* in the Julia set**. For the Julia set itself, we assign bright colors. The color assigned depends on the value of **z** after 256 iterations. For example, if the square of the modulus of **z** is greater than .6, but less than .7, the point z(0) is assigned a light red color. Hence the **colors in the Julia set indicate the value of the modulus of z at the end of 256 iterations.**

To see the second demonstration of the same equation, but with this alternative color mapping, be sure Java is enabled on your web browser and click here

So, from the complex logistic equation, a dynamical system, we have created a fractal. The border of the Julia set is determined by **k** in the equation, and this border was created in a working Euclidean space of 2 dimensions, has a topological dimension of 1, but has a Hausdorff dimension that lies between these two numbers.

Meanwhile, we have passed from mathematics to art. Or maybe the art was there all along. We just had to learn how to appreciate it.

Notes

[1] This is the stock market equivalent of the Interest Parity Theorem that relates the forward price F(t+T) of a currency, T-days in the future, to the current spot price S(t). In the foreign exchange market, the relationship can be written as:

$$F(t+T) = S(t)\,[1 + r\,(T/360)]/[1+r^*(T/360)]$$

where r is the domestic interest rate (say the dollar interest rate), and r* is the foreign interest rate (say the interest rate on the euro). S is then the spot dollar price of the euro, and F is the forward dollar price of the euro.

We can also use this equation to give us the forward value F of a stock index in relation to its current value S, in which case r* must be the dividend yield on the stock index.

(A more precise calculation would disaggregate the "dividend yield" into the actual days and amounts of dividend payments.)

This relationship is explored at length in Chapter 5, "Forwards, Swaps, and Interest Parity," in J. Orlin Grabbe, *International Financial Markets*, 3rd edition, Prentice-Hall, 1996.

[2] The definitions here follow those in Gennady Samorodnitsky and Murad S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman & Hall, New York, 1994.

[3] This is Theorem VI.1.1 in William Feller, *An Introduction to Probability Theory and Its Applications*, Vol 2, 2nd ed., Wiley, New York, 1971.

[4] If Y = $\mathbf{X/n^{1/\alpha}}$, then for **n** independent copies of Y,

$$Y_1 + Y_2 + \ldots + Y_{n-1} + Y_n \sim \mathbf{n^{1/\alpha}\, Y = n^{1/\alpha}\, (X/n^{1/\alpha})} = \mathbf{X}.$$

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from *The Laissez Faire City Times*, Vol 3, No 29, July 19, 1999

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