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Suppose you are playing a game in which you have some sort of edge. This edge can consist of any combination of probability and odds such that each bet has a positive expectation. Now you ask yourself, "What fraction of my bankroll should I wager on this game?" If your answer is, "The amount that if I keep repeating this same strategy over a long period of time, my bankroll increases at its maximum rate.", then your answer is to bet "Kelly".

To see how we can find such an answer, we should look at it from the beginning. Suppose you bet a fraction f (Bet size = f*B, odds = v-to-1, amount won = v*f*B, amount lost = f*B)

If you win, your new bankroll B' is going to equal your old bankroll plus your winnings:

Win: B' = B + v*f*B = (1+v*f)*B

If you lose, your new bankroll will equal your old bankroll minus your losses:

Lose: B' = B - f*B = (1-f)*B

Let's take a short timeout to make sure these funny looking equations make sense by looking at an example (I'm a huge fan of concrete examples).

Say you have $100, and you bet $20, getting 2-to-1. For the sake of future considerations, we'll assume the game is a fair coin flip (but the need for the probabilities does not come into play for awhile yet). If you win this bet, you win $40, raising your bankroll to $140. The equation above shows this correctly:

B = $100, f = 1/5 = 0.2, v = 2: B' = (1+v*f)*B = (1+2*0.2)*($100) = $140

The "Lose" equation works in a similar manner.

Now, suppose you play the game multiple times. Assuming no "pushes" (which I will be assuming throughout), then the "new" bankroll (which we have called B') after a game is played becomes the "old" bankroll for the next game to be played. We are assuming the game and the odds offered don't change, and that we do not choose to change our strategy (the fraction of bankroll to be wagered), so the equations given remain the same for each game, with the previous game's B' becoming the next game's B. Your result after winning twice in a row would be:

B after two wins = (1+v*f)*[B after one win] = (1+v*f)*[(1+v*f)*(starting B)]
B after two wins = [(1+v*f)^2]*(starting B)

A check: Suppose in the example above you won again, employing the same strategy. Your new bankroll was $140, so you wagered 1/5 of it ($28) and won at 2-to-1, for a total win of $56. Now your new bankroll is: $140+$56 = $196. Plugging into the equation gives the same result:

B after two wins = [(1+v*f)^2]*(starting B) = [(1+2*0.2)^2]*($100)
B after two wins = [(1.4)^2]*($100) = [1.96]*($100) = $196

The two losses results work out the same way. What about a win and a loss, you ask? You just multiply the starting bankroll by the "lose factor" (1-f), and the "win factor" (1+v*f), and the result is the new bankroll. Note that it doesn't even matter if you lost first or won first. Back to our example:

You won the first game and lost the second: $100 + $40 = $140 after first game, then lost $28 (you wagered 1/5 of it) in the second game for a total of $140-$28 = $112. If you lost first, you have $100 - $20 = $80 after the first game, and then you wager 1/5th of $80 (=$16) at 2-to-1 in the second game and win, for a total of $80 + 2*($16) = $112. Same amount as if you win first. [BTW, this less than obvious fact may affect retirement planning for those of you thinking about Roth IRA's vs. traditional ones. I won't digress any further on this topic.]

Using the equation gives the same result:

B after 1 win and 1 loss = (1+v*f)*(1-f)*(starting B) B after 1 win and 1 loss = (1+2*0.2)*(1-0.2)*($100) = (1.4)*(0.8)*$100 = $112

Okay, so lets say we have won "w" times out of n total games. This means we have lost n-w times (since we assumed no pushes). To find the new bankroll, we need to multiply the starting bankroll by (1+v*f) a total of w times, and multiply it by (1-f) a total of n-w times. In other words, our "new bankroll equation" has gotten much more complicated:

B' = [(1+v*f)^w]*[(1-f)^(n-w)]*B

The first factor in brackets is just the "win factor" multiplied by itself w times, and the second is the "lose factor" multiplied by itself n-w times. The two "B's" in this equation are not important, so we will instead look at just the factor multiplying B, as this is what determines the bankroll's growth (or lack thereof):

B'/B = [(1+v*f)^w]*[(1-f)^(n-w)]

This gives us the factor that the bankroll has changed from the beginning (after n games). We want to look at the bankroll's growth rate PER GAME, so if we call the average-per-game-factor "y", then after n games, the bankroll has grown by a factor of y^n. The average-per-game-factor is then found to be:

B'/B = y^n

y = (B'/B)^(1/n) = {[(1+v*f)^w]*[(1-f)^(n-w)]}^(1/n)

Again we return to our example. After 2 games where we won 1 and lost 1, we have gone from $100 to $112. This is an increase of a factor of 1.12 for 2 games. On average, this is an increase PER GAME of a factor of sqrt(1.12) = 1.058 . In other words, if in the game described, we win the same number that we lose (i.e. we assumed way back when that the probability of winning is 1/2), on average we will increase our bankroll each game by a factor of 1.058. Now here's the important part:

*** If we employ a different strategy (risk a different fraction "f" of the bankroll), then this factor will also change. We seek to find the f for which this factor is a MAXIMUM. ***

Now finding the value of f for which this function peaks is no small matter. It involves calculus. If this intimidates you, I invite you to jump down to below the second set of "*'s" to see the answer. I include the calculus for the math weenies that may find it interesting...

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The value of f for which y(f) is a maximum is the same value for which ln[y(f)] is a maximum, so we can equivalently seek to maximize:

z(f) = ln[y(f)] = (1/n)*[w*ln(1+v*f) + (n-w)*ln(1-f)]

The derivative is:

$dz/df = (w/n)*v/(1+v*f) - [1-w/n]/(1-f)$

Setting equal to zero and finding f gives:

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$f = [p*(v+1)-1]/v$, where $p = w/n$.

Note that in the long run, the fractional number of times you will win a game (w/n) equals the probability of winning a single game, so p = probability of winning.

Okay, this is our answer. Given that your chance of winning is p, and that you are receiving v-to-1 odds on your bet, then the fraction of your bankroll that you should wager to maximize your rate of bankroll growth is the f given above. This value can also be plugged back in above to find out what the maximum growth rate actually comes out to be.

Let's try it for the game we've been using as an example. We have a probability of p=0.5 of winning, and odds of v=2, so the fraction of our bankroll we should risk each time is:

$f = [p*(v+1)-1]/v = [0.5*(2+1)-1]/2 = 1/4$

$y = [(1+v*f)^p]*[(1-f)^(1-p)] = 1.061$ (recall p=w/n)

The bankroll increases an average of 6.1% over its preceding amount every game. [Using the "rule of 72" familiar to bankers and investors, this means the bankroll will double roughly every 12 games.]